Paul Hertz—Polymath

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Abstract. Paul Hertz was a significant contributor to Hilbert's school at Göttingen and a polymath whose work is lauded in the literature of several fields of mathematics and theoretical physics but has never been subject of a biography that encompasses his entire range of achievements. I have collected material for one for many years but no longer have time to write it. These notes summarise much of his background and many of his achievements in a hope they will encourage others to document them in greater depth. There is little personal biographical data available but substantial circumstantial evidence that he was heavily influenced by his admiration for his third cousin, Heinrich Hertz, and following his career path in his own. This, if correct, explains the diversity of research topics to which he contributed because Heinrich exhibited the same diversity, but that is often not appreciated because his major breakthrough in demonstrating the physical reality of the propagation of radio waves in free space overshadows his other research.

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1. Introduction

Paul Hertz was a mathematician and theoretical physicist in Germany in the early 1900s who played a significant role in the development of science at a time when it was going through a significant transition with the advent of quantum mechanics, special and general relativity theory, concerns about the security of its mathematical foundations, and the world went through two world wars in which new scientific developments played a major part. Much has been written about many of the major scientists of that time but very little about Paul, partly because his contributions have been slowly recognised in diverse disciplines and partly because little information about him was captured in his lifetime.

These notes are an attempt to document his contributions to science and the background to his academic research, and suggest some influences on his career for which there is only circumstantial evidence but which account for its dynamics, in particular that he admired his famous cousin, Heinrich Hertz and his philosophy of science and took him for a role model. Hence, Heinrich's relevant contributions will also be noted.

2. The expansion of support for science in Germany to facilitate industrialisation

Much of the advance in science was centred in Germany because the Kaiser had approved massive funds being made available to German Universities to support the industrialisation of German industry [11]. Under the leadership of Friedrich Althoff, the universities of Prussia, the dominant kingdom within the German Empire, were transformed into some of the best-structured and well-resourced academic institutions in the world [11].

Two universities that benefited greatly were the University of Berlin where Paul's cousin Heinrich Hertz studied and the University of Göttingen where Paul studied.

2.1. The University of Berlin, Helmholtz and Heinrich Hertz

Hermann von Helmholtz held the Chair of Physics at the University of Berlin[1] and was Germany's leading scientist. He worked closely with Althoff to transform his university into a role model for the institutions that the Kaiser wished to create. Helmholtz had championed Maxwell's equations in Germany in the 1870s where they had been initially rejected in the 1860s. He had developed a specialised form of the wave equations that were more understandable, but was still able to predict the most interesting consequences such as electromagnetic radiation.

In 1879, Helmholtz encouraged the Prussian Academy of Sciences to offer a 100-ducat prize for experimental proof of Maxwell's predictions. He encouraged his student, assistant, mentee and potential successor, Heinrich Hertz, to carry out the research, but Hertz felt an apparatus sensitive enough to test the theory could

[1]Now Humboldt University of Berlin.

not be built. No-one won the prize. Helmholtz continued to encourage and mentor Hertz, finding him good positions. In 1885 he was appointed Professor of Physics at the Technische Hochschule in Karlsruhe and found equipment that could be used to create and detect radio waves. He also measured the speed of propagation as the same as that of light.

In 1892 Hertz's collected papers were published [25] and in the introduction he raises the question of what is Maxwell's physical theory, concluding "When asked, "What is Maxwell's theory?", I could therefore think of no briefer or more precise answer than this: Maxwell's theory is the system of Maxwell's equations."[2]

This example of a mathematical description of a range of major phenomena that was not grounded in physical intuitions led Hertz to his next major achievements: a re-axiomatisation of Newtonian mechanics that addressed the concerns of Kirchhoff, one of his teachers and mentors at Berlin, that metaphysical notions such as force should be treated as mathematical fictions rather than causal principles—that mechanics should simply and completely describe natural motions without searching for their underlying causes [44, p.v].

This led to his *picture-conception* of theories which had a profound effect on Hilbert, Weyl, and Ramsey, and motivated Wittgenstein's Tractatus [51]. Paul Hertz endorsed it for similar reasons as Heinrich, that it corresponded to his experience of Maxwell's and Gibbs equations and, in his doctoral studies,he had seen how correct mathematics applied to an incorrect physical picture of the electron led to incorrect predictions (§3.2).

2.2. The University of Göttingen and Hilbert

The George Augustus University of Göttingen, founded in 1734 by the Elector of Hanover (who was also King George II of Great Britain), was one of those to benefit. In 1886 Felix Klein was appointed full professor of mathematics and began his lifelong effort to transform the university into a global center of excellence for mathematics and applied physics. He worked closely with Althoff to secure the funds to attract top mathematicians and physicist, one of whom was David Hilbert.

Klein had published Hilbert's *basis theorem* in the *Mathematische Annalen*, writing to him, "I do not doubt that this is the most important work on general algebra that the Annalen has ever published." In a report to Althoff in 1890, he described Hiilbert as "the rising man" in mathematics but did not manage to recruit him until 1895, later needing Althoff's help to retain him when in 1902 Hilbert was offered the Mathematics Chair at the University of Berlin—the counter-offer was a full professorship at Göttingen for Hilbert's friend Minkowski.

Under Klein and Hilbert, Göttingen developed an international centre of excellence in mathematics and physics [63] providing what has been described as "a richer picture of mathematics" [64], constituted by deep abstraction, conceptual

[2] Auf die Frage ,.Was ist die Maxwellsche Theorie?" wusste ich also keine kürzere und bestimmtere Antwort als diese: Die Maxwell'sche Theorie ist das System der Maxwell'schen Gleichungen.

rigour, and a close unity between pure mathematics and theoretical physics. Paul Hertz was steeped in, and contributed to, that for most of his career.

3. Paul Hertz's career

German universities in Hertz's time had a different procedures than they do today. Students studied directly toward a doctorate in their discipline without completing a lower degree first. Over four years, they generally took appropriate courses at several universities whilst seeking the opportunity to undertake research on a topic of interest under the supervision of a knowledgeable advisor at a university with appropriate facilities, and then present a dissertation describing their research and courses for examination. The fees the students paid for the lectures and examinations were paid to those giving the lectures or conducting the examinations.

Hertz's curriculum vitae at the end of his dissertation is the best historic record of his early history and academic education [27, p.82]:

> I, Paul Hertz, was born on July 29, 1881, in Hamburg, the son of the attorney Dr. iur. Eduard Hertz and his wife, Elisabeth, née Goldschmidt. From Easter 1891 to Easter 1900, I attended the Wilhelm-Gymnasium in Hamburg. At Easter 1900, I enrolled at the University of Heidelberg to study mathematics, physics, and philosophy, remaining there for two semesters. During the summer semester of 1901, I studied in Göttingen, and from Michaelmas 1901 until Easter 1903, I studied in Leipzig. At Easter 1903, I returned to Göttingen.
>
> During this period, I attended lectures, seminars, and practical courses led by the following professors and lecturers:
>
> In Heidelberg: Aschaffenburg, Curtius, Kuno Fischer, Klaatsch, Königsberger, and Quincke.
>
> In Göttingen: Abraham, Blumenthal, Hilbert, Klein, Minkowski, G. E. Müller, Riecke, Voigt, and Zermelo.
>
> In Leipzig: Boltzmann, Hausdorff, Hölder, Wiener, and Wundt.
>
> I hereby express my warmest thanks to all these esteemed teachers of mine. I am especially indebted to Dr. M. Abraham, who provided me with excellent guidance throughout my entire course of study and who also suggested the topic for the present work.

Together with the Universities of Berlin and Munich, the universities where Hertz studied were the top 5 in Germany in 1900. Figure 1 shows their geographic locations and the train travel time between them and his family home in hours. They were all reasonably accessible given the excellence of the German railway system, but they also had first class mathematics and physics department where Hertz studied and that choice of disciplines may well have been affected by his admiration for Heinrich Hertz.

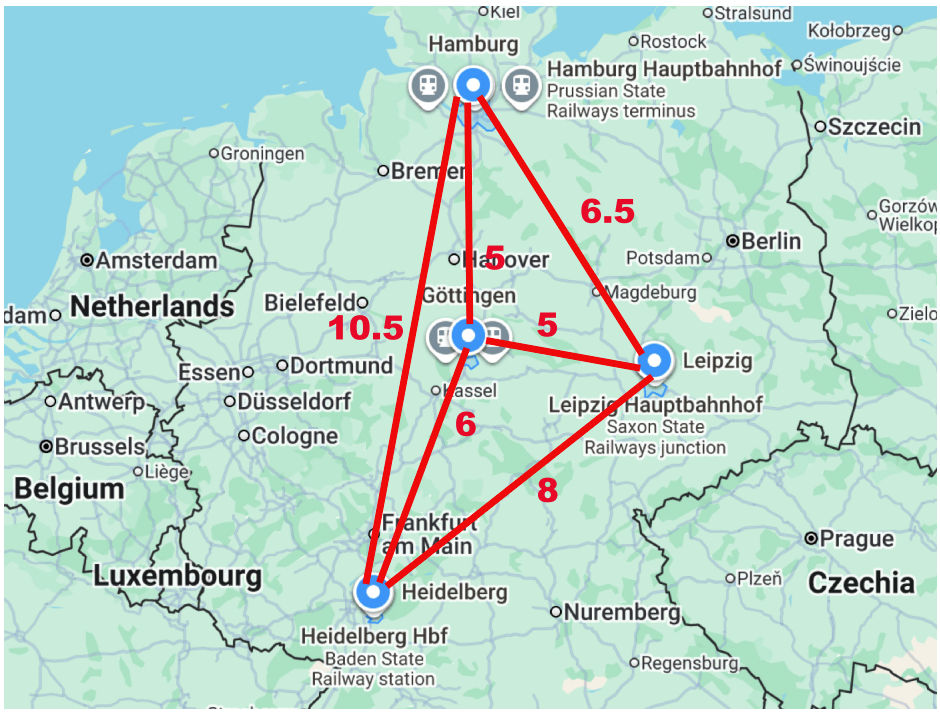


FIGURE 1. Paul Hertz's home and university locations and train travel times in 1900 (map courtesy of Google)

3.1. Paul Hertz's relationship to Heinrich Hertz

In studying Paul Hertz, I came across circumstantial evidence that he was influenced in his career by his admiration for his third cousin, Heinrich Hertz, a scientist and philosopher of international renown [4]. I have not seen this relationship noted in the literature and will use it to structure this exposition but if the conjecture is false it should not detract from the utility of the remaining data about Hertz.

Paul came from a prominent Jewish family and his genealogy is available in detail for many generations. Figure 2 shows a section that includes both Heinrich and Paul. Paul never mentions Heinrich directly but he named his second son Rudolf Heinrich after him and his first two logic papers focus on the role of *ideal elements* using as an example the notion of *force* in mechanics which Heinrich made the subject of his last book that successfully re-axiomatised Newtonian mechanics without using the notion of force [26]. These scraps of evidence suggested to me that Paul might have read Heinrich's major works at least and might have been influenced in his choice of career by his admiration for him. We do not know if they ever met and Heinrich died prematurely when Paul was twelve so it would not have been a deep discussion.

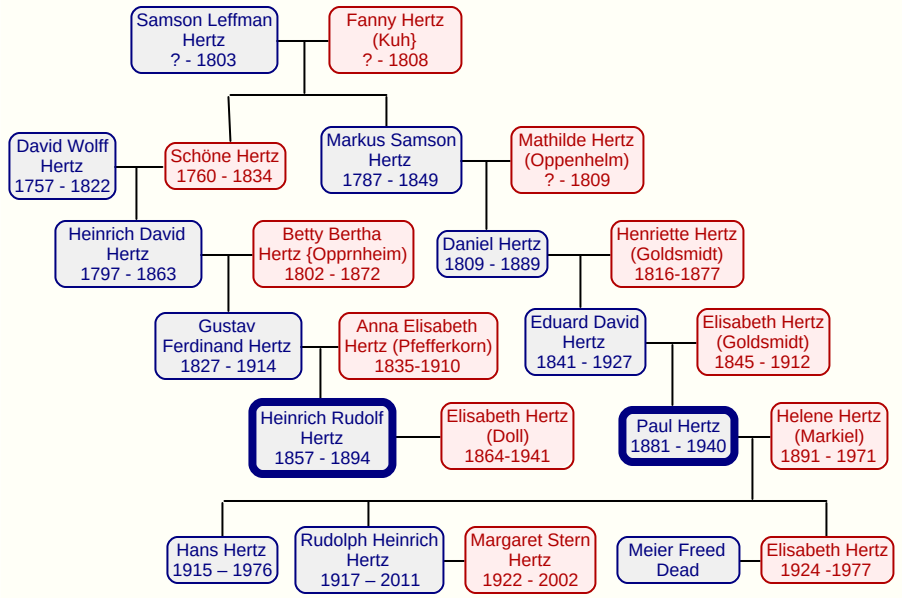


FIGURE 2. Heinrich and Paul Hertz family tree (based on *Neue Deutsche Biographie)*

Their early research involved application of Maxwell's equations, Heinrich to electromagnetic wave propagation, Paul to the Abraham model of the electron. Their later research involved mechanics, Heinrich's to Newtonian mechanics, Paul's to Gibbs' thermodynamics. Heinrich made significant improvements to Maxwell's equations and Paul to Gibbs' laws; Heinrich also re-axiomatised Newton's laws; treating force as ideal object not a causal variable. Heinrich developed a significant philosophy of science depending strongly on logic, Paul developed a new philosophy of logic and saw philosophy as his mature area of research[9, p.3 §2]. None of this is conclusive, but a reasonable indicator that we might gain insights into Paul by studying Heinrich.

3.2. Paul Hertz's doctoral studies with Max Abraham

Paul Hertz's doctoral studies were registered at the University of Göttingen nominally under Hilbert but actually under the Privatdozent, Max Abraham, a cousin of Paul [38]. Abraham was a brilliant theoretical physicist who specialised in Maxwell's equations and had developed a model of the electron as a rigid sphere moving through the ether [23]. Paul's contribution was to explore the consequences of accelerating an electron to speeds approaching that of light by rigorous application of Maxwell's equations. He was awarded his doctorate on 28-July-1904 [27].

The nature of the electron was one of the unsolved problems of electrodynamics and for the 3-month Göttingen summer session of 1905 Hilbert organised a weekly electron seminar to explore the state of the art using surveys by Lorentz and Poincaré to frame the discussion. The seminar was led by Minkowski, Hilbert, Wiechert, and Herglotz. Abraham presented his theory and Hilbert presented Paul Hertz's results which he had recently examined.

There were three primary contenders for an electron model, Abraham, Bucherer–Langevin and Lorentz. As empirical data was accumulated, from 1901-1906 it favoured Abraham, but then more refined experiments showed it false, and later the Bucherer–Langevin, eventually culminating in Dirac's quantum electron and quantum field theory.

Paul's reputation as a theoretician was not affected—his math was impeccable and faithfully derived the consequences of physical intuitions that proved wrong, an imperfect picture. Interest continues today into his research on the relativistic dynamics of a charged sphere [78, 79].

3.3. Paul's post-doctoral activities after his doctorate

Paul reports in a cv submitted to the University of Göttingen in about 1912 that between 1904-1912 he was in Paris, Göttingen, Berlin and Tübingen and the Netherlands, where he went in the summer of 1911 to work with Lorentz. He went to Heidelberg in 1907/1908 as an assistant to Phllip Lenard who had been Heinrich Hertz's assistant, and received the 1905 Nobel Prize in Physics for his research that extended Heinrich's on cathode rays.

While there in 1908 or 1909 he completed his habilitation (license to teach) with a dissertation on the String Galvanometer [28] under the supervision of Lenard and Leo Königsberger. In 1912 he moved back to Göttingen as a Privatdozent [9, p.4]. It has been suggested that Lenard's anti-semitism may have been a problem but that developed later, or his demand that all work in his laboratory was attributed to him [58] or it may have been simply that Hilbert offered him a place. In a letter to Dingler dated 26-Dec-1914, Hilbert mentions Paul Hertz, Bernstein, and Grelling as his closest associates on foundational issues [**?**, p.322n.4].

He was made professor extraordinarius (associate professor) there in 1921. In 1933, as a Jew, he was barred from teaching. He moved to Switzerland and then to Prague (working at the German University of Prague starting around 1936). He emigrated to the United States in 1938 or 1939 but never managed to establish himself professionally afterward, although in 1939 Yale invited him to give a lecture for the centenary celebrations of Gibbs which was published posthumously /citeHertz1956. He died in Philadelphia in 1940 at age 58.

3.4. Hertz's contribution to numerical analysis

Hertz's most cited paper [29] states "For some questions in molecular physics, it is important to be able to estimate the average distance between two neighbouring molecules" and presumably relates to some issues in statistical mechanics where he was about to publish a major treatise the following year. He solved the problem

under the assumption that the molecules were randomly distributed in space, but his algorithm it is applicable to computations in many disciplines and became a contribution to. numerical analysis.The distribution became known as the *Hertz distribution*, a mathematical model of pure randomness in space. The paper had 347 citations on Google as of Nov-2026, mainly in recent times, because of the increasing use of computers in diverse disciplines. :

3.5. Hertz's contributions to Gibbs' statistical mechanics

Hertz published a major paper in 1910 (in two parts) and an encyclopaedic one in 1916 that established him as a world-class expert in Gibbs' statistical mechanics, not only as an expositor but also an innovator. It is not clear when and where he did the underlying study. However, Wilhelm Ostwald translated Gibbs's *On the Equilibrium of Heterogeneous Substances* into German in 1892 as *Thermodynamische Studien*. Yale University included his *Elementary Principles in Statistical Mechanics* in the 24-volume *Yale Bicentennial Publications* series gifted to major European academic institutions [22]. Max Abraham presented the book to the *Mathematische Gesellschaft zu Göttingen* on 1st and 8th July 1902, and also published a review [21]. Lorentz began lecturing on Gibbs thermodynamics in 1905 [60].

Thus, Hertz had many opportunities to become aware of, and compare Gibbs' more general abstract approach to that of Boltzmann's lectures when he was in Leipzig. He could see ways in which he could improve Gibbs' theories and also realised that Einstein's 1902-4 papers on thermodynamics that had similar premises to Gibbs had errors.

3.5.1. Hertz's use of a 'delta function' to show two bodies sharing energy have the same temperature. Hertz greatly simplified Gibbs proofs that two bodies sharing energy have the same temperature by the use of what we would now call a *delta function*. It is sometimes assumed that this was a heuristic until the theory of distributions was developed by Laurent Schwartz in the late 1940's, but Hertz's argument is rigorous under explicitly stated conditions (which Hertz himself flags as partly assumed): the peak must capture all but an arbitrarily small fraction of the integral; the other factors must stay bounded and not vary too fast near it; and the separation theorem is a high probability (like much of thermodynamics with no Maxwell's demons around!).

It should be noted that Dirac's later use of a delta function in the quantum mechanical context was independent and for a different purpose, to handle continuous spectra in quantum mechanics, normalising position and momentum eigenstates, and writing completeness relations. They need different formalisations in terms of the modern theory of distributions.

3.5.2. Hertz's refining Gibbs' definition of entropy. Hertz substantially refined Gibbs' definition of entropy such that later empirical studies of its application to quantum mechanical system have renamed it the Hertz entropy, although this has been contentious. He did not invent a new formula for entropy. Gibbs provided

three possible definitions of entropy. Hertz rejected the *canonical ensemble* that Gibbs preferred arguing that it has no physical meaning, calling it at most an analytical device, grounded the *microcanonical* ones in the time behaviour of a single system, rather than a fictitious collection of independent copies (following Einstein, and a concept Gibbs had mentioned but not used).

Both these objections have the same basis as Heinrich Hertz's objections to Newton's use of the notion of force, that they are mathematical fictions, or ideal elements, and do not correspond to a physically meaningful entity. They show him as a theoretical physicist, not just a mathematician, and validates those who now call it the Hertz entropy.

He also selected lnV for entropy over Gibbs's other alternative on the physical ground that it yields exact equipartition and the Clausius' virial theorem for any number of degrees of freedom, proved that the associated temperature behaves correctly under thermal contact, and showed that V is an adiabatic invariant.

Those who say he merely chose one of Gibbs's definitions [39] are right about the formula but miss these other contributions, which are why the volume entropy now bears his name. I conjecture that his disclaimer that he was just choosing one of Gibbs' option was just politeness to a senior colleagoe—he was actually making profound changes.

The full significance has only been appreciated in the current era when the quantum version, *quantum Hertz entropy*, being argued to be more appropriate than the von Neumann one, extending to sparse, non-equilibrium systems [43, 12], such as those of quantum computers.

This is very significant for the current trend of unifying the micro-dynamics of the quantum world with macro-dynamics of the large-scale world. For example, that Clausius' formulation holds in both [43]. It doesn't really matter for applications but the unification of two domains so that patterns of inference are the same in both is attractive for many reasons and may result in deeper understanding of the universe.

There are also more abstract information-theoretic arguments for the emergence of a second law of thermodynamics in isolated quantum systems [57], and these provide interesting perspectives on how 'pictures' of the universe based on very general intuitions like Shannon relate to those based on 'physical' intuitions like those of the Hertz cousins.

Hertz would not have been surprised with the significance of Gibbs theories for quantum mechanics, and would have agreed with my 'deeper understanding' assessment. That is the theme of his very last paper [37], published posthumously in 1956, 16 years after his death. Paul was invited by Yale to give a lecture for the 1939 Gibbs memorial year, but was too ill to deliver it [6]. Bernays and Alfred Landé published and commented on it in a 10th anniversary issue of Dialectica [6].

3.5.3. Hertz's criticisms of Einstein's papers on thermodynamics. Einstein published a series of three papers on thermodynamics in 1902-4 that proposed very

similar mathematical frameworks for thermal equilibrium to Gibbs [21]. Hertz commented on this in his 1910 paper and also noted that Einstein relied on the unproved assumption that a system will never spontaneously transition from a more probable state to a less probable state, and critiqued the rigour of how Einstein connected mechanical ensembles to the thermodynamic definition of entropy and irreversibility.

Einstein sent him a letter suggesting they meet to discuss this and then sent a note to the *Annalen der Physik* stating that Hertz's first criticism was totally valid and stated that he would never have published his 1902–1904 papers had he known about Gibbs's work at the time [16]. The outcome was that the two developed a collegial friendship that had significant consequences for the development of the theory of general relativity, and that a derivative literature developed [56, 59, 21] discussing what Einstein contributed to thermodynamics.

3.5.4. A misattribution in a commentary on Hertz's role in the theory of distributions. The 1916 chapter for Weber and Gans' *Repertorium der Physik* [32] includes the material from 1910 but is more tutorial occupying 164 pages.

Unfortunately, Lützen uses the section on a delta function in this book [50, p193n14] to provide an assessment of Hertz's role in evolving the theory of distributions, but attributes it to Weber & Gans, the editors of the book[3]. He also omits to note that Hertz used the the function in his earlier 1910 paper, that he ensures his use is sound, and that his constraints play the role of Schwartz's test functions and exemplify the theory of distributions.

3.6. Hertz's contribution to Einstein's development of general relativity

The development of Einstein's theory of general relativity has been one of the greatest sagas of science, physics and mathematics throughout human history. While Einstein was undoubtedly the driver of the effort the difficulty of the task, both conceptually and mathematically, made it a team effort with many of the top mathematicians and physicists of the era contributing to clear the roadblocks along the route, particularly Hilbert [15, 65, 70, 40]. The 4-volume, encyclopaedic *Genesis of General Relativity* [62] provides a very detailed account of the saga, those involved and their contributions.

Those of Paul Hertz are documented in detail in *Genesis of General Relativity* in terms of the *Hertz expression* (62 occurrences), *Hertz condition* (16 occurrences) and *Hertz restriction* (16 occurrences), for the critical mathematical contributions that he made in his correspondence with Einstein. Hertz being part of Hilbert's team at Göttingen, who Einstein saw both as rivals and as partners, and his personal friendship with Einstein may have made him an acceptable go-between so it is difficult to know who to credit for this input, particularly that relating to Einstein's *hole argument* [41, 10], but his name has gone down in history.

[3]The error may be because the chapter which Hertz wrote is not headed by his name—the attribution is at the beginning of the book.

3.7. Hertz's contributions to modern logic

It has been widely recognised and documented that the basic principles of the *structural reasoning* associated with the Gentzen revolution in logic were present in Hertz's 1921-23 [33, 34, 35, 36] papers that Bernays gave to Gentzen to initiate his research [1, 2, 46, 47, 9].

What has been less studied and valued is the graphical notation that Hertz invented to express his theory and I will focus on that. It seems likely that this innovative notation may have been in response to Moses Schönfinkel's lecture to the Göttingen Mathematical Society on 7-Dec-1920 on *Elemente der Logik* where he criticised the unnecessary complexity of logical notations.[4]

Schönfinkel motivates his objectives by noting that "a variable in a proposition of logic is, after all, nothing but a token that characterises certain argument places and operators as belonging together; thus it has the status of a mere auxiliary notion that is really inappropriate to the constant, eternal essence of the propositions of logic"[5].

Hertz was present at his presentation, and may have already discussed these issues with him, but saw another way of addressing them, by writing logical formulas in ***two dimensions*** and leaving the meaningless place-marking tokens out. This is illustrated in Figure 3 where the symbolic inference pattern on the left defines the transitivity of the arrow using the usual quantified variables as meaningless place markers.

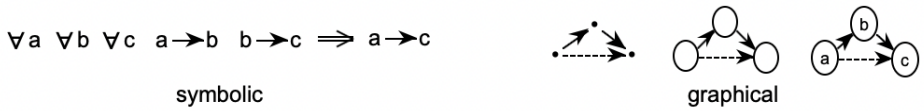


FIGURE 3. Symbolic and graphic inference patterns

The first graphical figure on the right is what Hertz invents to address Schönfinkel's concerns, reducing the number of symbols by 10 to 13 (16 symbolic, 3 or 6 on the graphic side depending on what you count). The second is the same as the first with the node dot big enough to contain a label if needed. Both are logically equivalent to the first. The third labels the nodes so that they can be referenced from another part of the net—dummy variables may be useful sometimes. Hertz also invents the use of a dashed line to represent the inferred links. Notice that the universal quantifiers are implicit in the assumption that any label can be placed in any empty (anonymous) node.

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[4]Wolfram has celebrated Schönfinkel's solution because it led to combinatory logic, establishing the first complete mathematical formalism for universal computation—16 years before Alan Turing's machines and Alonzo Church's lambda calculus [77].

[5]„die Veränderliche in der logischen Aussage ja nichts weiter als ein Abzeichen ist, um gewisse Argumentstellen und Operatoren als zusammengehörig zu kennzeichnen, und somit den Charakter eines bloßen, dem konstanten, *ewigen* Wesen der logischen Aussage eigentlich unangemessenen Hilfsbegriffes hat"[67, p.307]

Figure 4 shows Hertz's first examples of inference in his new notation. He used this notation in his first two publications on research on refactoring axioms using ideal elements to reduce their complexity.

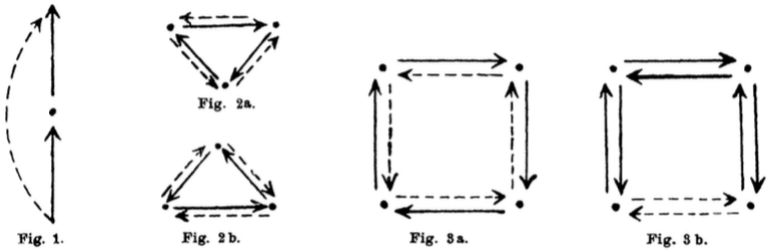


FIGURE 4. Paul Hertz's first example of 2-dimensional inference [33, p.155]

However, Hertz did not further pursue the graphical notation beyond his first two papers, and reverted to a symbolic notation in developing his logical system further.

Another member of Hilbert's team in Hertz's time that used graphical notation for logic and could have influenced Hertz was Heinrich Behmann. Paolo Mancosu has documented his activities in depth [52, 53, 54, 55] including his lectures to the Göttingen Mathematical Society Behlman presented on mathematical logic in 1914, once on Russell-Whitehead logic in 1916 and twice in 1917 [52]. He lectured to the Göttingen Mathematical Society on advances in modern logic, once in 1914, once on Russell-Whitehead logic in 1916 and twice in 1917 [52], some of which which Hertz may well have attended.

Behmann made very extensive use of graphical inference patterns for propositional logic, classes, and relations in his 1927 book *Mathematik und logik* [5]. As extensively analysed by Mancosu [53], he also wrote a paper on *constructivity* in logic, of whether a *reductio* proof can, subject to some well-defined restrictions, be replaced by a direct one. His purported proof that this was so was graphic using arrows similar to Hertz but with a different interpretation. However Gödel provided a counter-example and the paper was never published.

Bodner has recently published a wide-ranging assessment oof Hertz's contribution to proof theory, including much newly translated material from Behmann. Behmann was very critical of Hertz's logic papers, noting that they are logically trivial, representing what is already well known. An appendix to this paper addresses this criticism, illustrating how Hertz's graphical notation can be easily extended to provide very powerful foundations for modern logics.

His research was further developed by Gerhard Gentzen, a later member of Hilbert's group, and evolved into his *sequent calculus* that revolutionised the foundations of proof theory in modern logic [20, 7]. Gentzen made no use of diagrams in his publications, but does use arrow diagrams in his own notes in proving his Hauptsatz (Fig. 5).

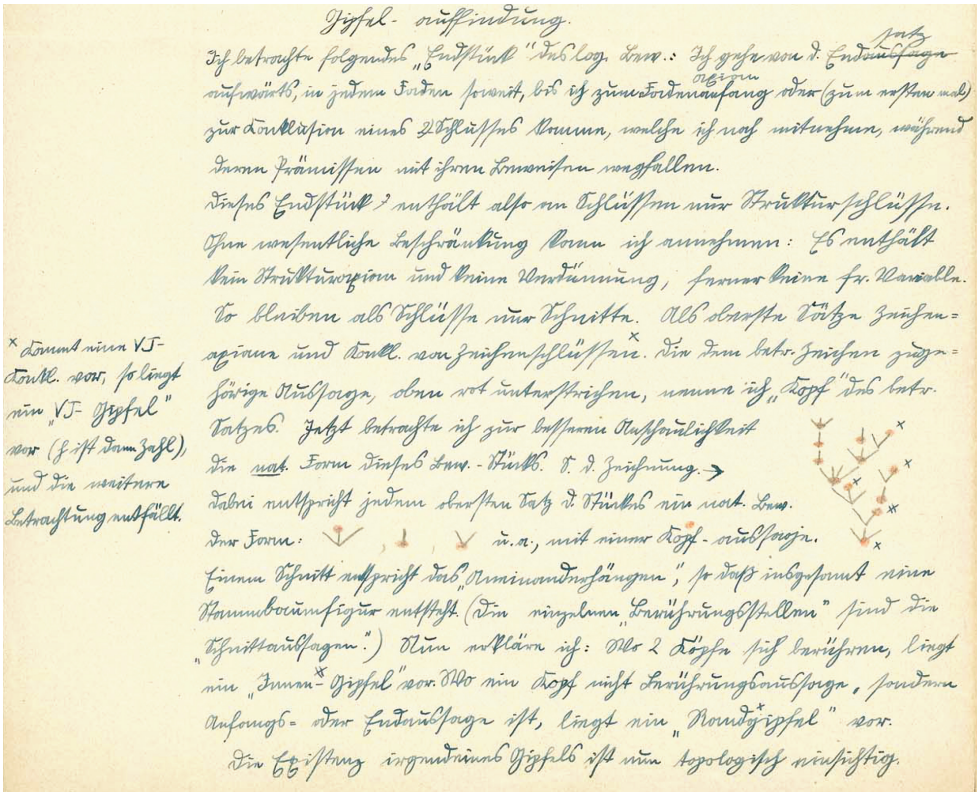


FIGURE 5. Gentzen's use of graphics inference patterns [74, p.177]
(Copy of bottom of page of kindly supplied by John von Plato)

4. Conclusions: The recognition of Paul Hertz's contributions

Little has been published about Paul's intellectual development and contributions as a whole, partly because little personal information is available and partly because the significance of his contributions was not recognised until after his death. The notes above are intended to identify them more clearly, and encourage others to investigate them in greater depth.

One early recognition is that in 1921, the centenary of Helmholtz' birth, Paul Hertz and Moritz Schlick were asked to edit and comment on a selection of Helmholtz' epistemological writings. Paul annotated his *On the Facts Underlying Geometry* and *Numbering and Measuring from an Epistemological Viewpoint.* The 1977 English edition notes that "Their edition of Helmholtz was of historical influence, comparable to the influence among contemporary mathematicians and philosophers of Hermann Weyl's annotated edition in 1919 of Riemann's great dissertation of 1854 on the foundations of geometry."

Another early recognition is Levi-Civita's 1927 [48, 49] exposition of the importance of Paul's 1910 proof that V is an adiabatic invariant in Gibbs systems [30, 31], and renaming the relevant theorems the *Gibbs-Hertz theorems*.

An 1909 work on the distribution of molecules resulted in a general numerical algorithm for *Hertz distributions* such that his paper attracts more citations every year and is now his most cited.

As detaled above, he is recognised in the general relativity literature for the critical mathematical contributions that he made in his correspondence with Einstein.

There has been gradual recognition that the basic principles of the *structural reasoning* associated with the Gentzen revolution in logic were present in Hertz's 1921-23 [33, **?**, 35] papers that Bernays gave to Gentzen to initiate his research [1, 2, 46, 47, 9]. Von Plato has provided the most detailed analysis of the role that what he calls "Hertz systems" played in Gentzen's research [74].

The problem with the term "Hertz systems", that is used in several other papers [2, 3, 72, 73], is that is already used in the Heinrich Hertz literature to reference his 'systems of mechanics' [66, p.679], 'systems of hidden masses', or just 'systems' [61, p.260], and so on. It would be good to see ones that linked him to Gentzen, such as the *Hertz/Gentzen revolution* or *Hertz/Gentzen structural reasoning*.

The graphical notion that he invented for logical inference structures for his first two logic papers were recognised by Dénes König when he wrote the first book on graph theory [45] as major contribution to the foundations of *transitive graph theory*. Arndt documented this and extended the notation to encompass multi-premiss axioms [3]. Gaines extended it in a different way to do that and express the rest of Gentzen's sequent calculus as graphical patterns [18]. Gentzen dropped this aspect of Hertz's work and it has yet to be fully explored.

Thus, aspects of his research are widely appreciated but in diverse disciplines that have no reason to be aware of all of it. But there is a coherence to it that this paper begins to address, and may be relevant to others assessing his historical role and significance.

Whether, the breadth and depth of his work merits the qualification of *polymath* may be controversial. What seems clear is that he was a superb theoretical physicist who valued precision and physical foundations and improved many of the early formulations of the major innovations in theoretical physics in his time.

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5. Appendix on Inference Patterns

This appendix analyses how Hertz's visual language for logic might been extended to cover more of logic and address Behmann's criticism. Arndt showed how his graphical notation could be extended to multi-premiss inference using hypergraphs [3], and Gaines did this also by introducing an additional arrow type [18]. This last paper covers most of the material relevant to this appendix, as does the longer draft paper from which it was drawn [19].

6. Inference Patterns

I have used the term *inference pattern* to describe the inference structures in Figures 3 and 4. The term is commonly used in many literatures [42, 71, 68, 24, 14, 13, 75, 76] but not defined, either formally or informally. It is assumed that the meaning is apparent from those of the component terms and the context of use. Hertz represented the notion as a visual pattern, a structure of nodes connected by links in which some of the links, if absent from the structure, may be inferred to be implicitly present[6]. Hence, if present, they may be removed without changing the interpretation of the structure since they are implicitly present [18].

The one major change I made since those papers was to define exclusion/opposition/disjoint/contrary as a symmetric link rather than as an asymmetric one that one then proves by reductio to be symmetric. That introduces an unnecessary circularity and, while I empathise with Sher's argument that the foundations of logic are necessarily circular [69] but see no reason to make it worse than it need be. I also see the attractiveness to Aristotle of the complentaritty of A belongs to all B and A belongs to no B.

Defining conjunction and disjunction in the pattern language is trivial and Paul could have done that. Smilary negation can be done siimply by coupling he contrary and its dual, the subcontrary and provides a better introduction than that of Gentzen.

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[6]In NETL, an early semantic network implementation, Fahlman used the evocative term *virtual copy* for such implicit links [17]