TIMS/Studies in the Management Sciences 20 (1984) 47–65
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FUNDAMENTALS OF DECISION: PROBABILISTIC, POSSIBILISTIC AND OTHER FORMS OF UNCERTAINTY IN DECISION ANALYSIS

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Decision analysis deals with problems of uncertainty arising from a variety of circumstances in a variety of forms. It requires mathematical tools to represent uncertainty and the strategies for operating with it – tools which are adequate to represent the key phenomena of decision analysis yet not so strong that they import artifacts of their own. Probability theory has been the major tool that has proved useful to date but in recent years fuzzy set theory has been put forward as a more applicable alternative in representing certain forms of uncertainty, notably that in possibilistic systems. In the early years of fuzzy set theory it was necessary to emphasize the new, to draw attention to the differences and their value, and this led to an over-emphasis in the literature on the lack of relation of fuzzy set theory to previous logics of uncertainty, such as probability theory. However, there are both fundamental differences and fundamental similarities and relationships. This paper gives a rigorous formulation of these relationships, bringing out the similarities and differences, and attempting to place these in the context of their significance in decision analysis. In particular it is shown that both probability theory and fuzzy set theory may be derived from the weaker underlying structure of a standard uncertainty logic, and that many of the key results relevant to decision analysis are derivable within this weaker framework.

1. Introduction

Decision: Latin *decidere* – a cutting away, a separation, the making of a distinction.

We are so used to the concept of making a decision that in coming to its analysis it pays to stand back a little and consider the roots of the term. Decision is fundamental to human life, and the processes underlying decision making are fundamental to all our knowledge and action, beliefs and values, science and technology (Gaines and Shaw [17]). Decision-making is part of any process of analysis and hence decision analysis itself presupposes decision, a recursive situation not uncommon in fundamental studies but one that gives little comfort.

The fundamentals of any subject area are always dangerous ground – often the more we look for the rationale behind what we are doing, its legitimation,

the more remote it becomes until eventually we are driven to the conclusion that there is none. Whether we decide that all is illegitimate and despair, or that all is legitimate and rejoice, or vice versa, it is unlikely that we shall find satisfaction in looking too deeply into the foundations of our activities. A very good model for human decision-making is that we decide, and then rationalize the decision afterwards. Perhaps that is also a good model for a scientific discipline, that we make it work, and how we rationalize the way we do it, the foundations, are fairly arbitrary.

However, there do come times when our current paradigms seem to have lost their cutting edge, when the very methodology of our subject area which has worked so well in advancing it seems unable to progress any further. There are times in the development of any discipline when an excursion into the foundations seems necessary in order to generate new tools to supplement the old in areas where they are proving to be inadequate. In 1962 Lotfi Zadeh suggested the need for such a re-evaluation of the foundations of system theory in the context of biology:

> There are some who feel... the fundamental inadequacy of the conventional mathematics – the mathematics of precisely-defined points, functions, sets, probability measures, etc. – for coping with the analysis of biological systems, and that to deal effectively with such systems, we need a radically different kind of mathematics, the mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions. Indeed the need for such mathematics is becoming increasingly apparent even in the realm of inanimate systems [35].

He went on to propose [36] a theory of *fuzzy sets* which was applied to decision theory (Bellman and Zadeh [4]), and over the next decade was taken up in a very wide range of disciplines from quantum physics and biology to computer and management sciences (Gaines and Kohout [16]). It also proved its practical worth in the solution of engineering control problems previously regarded as intractable (Mamdani and Assilian [24], Mamdani and Gaines [25]).

Zadeh's fuzzy set theory has very close links with the non-standard multivalued logics of Lukasiewicz (Rescher [29]), but it has gone beyond this in two distinct directions: first, in developing a linguistic semantics for the logic in terms of *fuzzy hedges* (Zadeh [37]), and secondly in its wealth of practical applications. The historians will probably account for fuzzy set theory as the development of semantics for Lukasiewicz logic that gave it a practical importance previously only possessed by the standard predicate calculus. It did not actually happen that way, but history is itself subject to revision through rationalization.

To see the significance of this work for decision analysis it is worth going back to the definition at the head of this paper and note that every decision is in essence what the Spencer Brown has termed *making a distinction*, and emphasizes as fundamental to all our knowledge and action:

> The theme of this book is that a universe comes into being when a space is severed or taken apart... By tracing the way we represent such a severance, we can begin to reconstruct, with an accuracy and coverage that appear almost uncanny, the basic forms underlying linguistic, mathematical, physical and biological science, and can begin to see how the familiar laws of our own experience follow inexorably from the original act of severance (Brown [6, p.v]).

Brown goes on to develop a "calculus of distinctions" which may be interpreted as standard propositional calculus (Schwartz [31]), but which also subsumes a variety of non-standard logics through other interpretations (Kohout and Pinkava [20]). These include Lukasiewicz logic as one of the forms.

These other interpretations arise when we reject the binary, black and white distinctions that we generally presuppose, that our terms of reference, our data, our measurements, our plants, and so on are well defined. Non-standard logics arise when we accept the *uncertainty* of our distinctions and note that definiteness in distinction is not available to us. In logic this occurs classically as the problem of the *borderline case*, the rose that is neither red nor not-red but somewhere in the excluded middle between the two (Sanford [30]). It is the basis of the *antimony of predication* that Russell discovered in Frege's foundations of arithmetic at a time when they seemed complete (Kneale and Kneale [19]): that every predicate, every distinction, does not serve to define a set – the axiom of comprehension that assumes that unrestricted predication leads to paradoxes in set theory. Zadeh's choice of Lukasiewicz logic as a foundation for his theory of fuzzy systems was vindicated in its fundamentals ten years later by the work of Maydole [26] and White [34]: the former showing that most non-standard logics also give rises to paradoxes in set theory, and the latter that Lukasiewicz logic does not. It remains to undertake the mammoth task of re-building arithmetic on logical foundations that are difficult for us who are over-tutored in the standard results and import them too readily when they are not legitimate (Ackermann [1]).

However, the practical vindication of fuzzy systems theory is probably of even greater relevance to decision analysis, that there are now enough applications of fuzzy reasoning in action to indicate something worthwhile. They perhaps also indicate a puzzle in that we already had techniques to deal with uncertainty. What is new, and how does it relate to the old? In the early years of fuzzy set theory it was necessary to emphasize the new, to draw attention to the differences and their value, and this led to an over-emphasis in the literature on the lack of relation of fuzzy set theory to previous logics of uncertainty, such as probability theory. However, there are both fundamental differences and fundamental similarities and relationships. This paper gives a rigorous formulation of these relationships, bringing out the similarities and differences, and attempting to place these in the context of their significance in decision analysis.

In particular, it is shown that both probability and fuzzy logics may be

derived from a weaker logic that is axiomatized as Lukasiewicz infinitely valued logic but is not truth-functional. Probability logic has the added axiom of the law of the excluded middle, and fuzzy logic has the added axiom of truth functionality. However, key results for decision analysis may be derived within the weaker logic and hence apply to both probability and fuzzy logics. Some paradoxes of classical logic are used to derive these results, including the generation of numerical truth values, probabilities, degrees of membership, decision rules and arithmetic means, as features of all the logics considered.

2. Classical, probabilistic and fuzzy logics

If the links between classical logic, probability logic and fuzzy logic are to appear clearly, so that similarities and differences are transparently obvious, it is useful to develop all three through parallel paths until the essential divergences are apparent. This section develops a logical system through a valuation on a standard lattice of propositions which corresponds to the classical propositional calculus if the valuation is binary, but to both probability and fuzzy logics if it is multivalued in the unit interval. The final divergence between probability and fuzzy logics is made through the addition of differing simple and natural requirements which the logic might additionally meet. However, it is noted that most of the key features and theorems of both logics are available before this divergence – *features and theorems significant for many applications are independent of the particular logic used.*

2.1. A standard uncertainty logic (SUL) as a valuation of a lattice

Let $L(X, F, T, \vee, \wedge)$ by the free lattice generated by a set of elements, X, under the two (idempotent, commutative) semigroup operations, $\vee$ and $\wedge$, with maximal element T and minimal element F, i.e. L satisfies:

$$\begin{aligned}
&(\text{P1}) \quad \forall x \in L, && x \vee x = x \wedge x = x,\\
&(\text{P2}) \quad \forall x, y \in L, && x \vee y = y \vee x, \quad x \wedge y = y \wedge x,\\
&(\text{P3}) \quad \forall x, y, z \in L, && x \vee (y \vee z) = (x \vee y) \vee z, \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z,\\
&(\text{P4}) \quad \forall x, y \in L, && x \vee (x \wedge y) = x, \quad x \wedge (x \vee y) = x,\\
&(\text{P5}) \quad \forall x \in L, && x \vee T = T, \quad x \wedge T = x, \quad x \vee F = x, \quad x \wedge F = F,
\end{aligned}$$

the idempotent, commutative, associative and adsorption postulates, together with a definition of the minimal and maximal elements (Birkhoff [5, p. 18]). The usual order relation may also be defined:

$$(\text{P6}) \quad \forall x, y \in L, \qquad x \leq y \leftrightarrow \exists z \in L : y = x \vee z.$$

Now suppose that every element of L is assigned a "truth-value", such as a "probability" or "degree of membership", in the closed interval [0, 1] by a continuous order-preserving function $p: L \to [0, 1]$ subject to:'

$$\begin{aligned}
&(P7) && p(F)=0, \quad p(T)=1,\\
&(P8) \quad \forall x,y \in L, && x \le y \to p(x) \le p(y),\\
&(P9) \quad \forall x,y \in L, && p(x \vee y)+p(x \wedge y)=p(x)+p(y),
\end{aligned}$$

i.e. p is a continuous, order-preserving *valuation* on L (Birkhoff [5, p. 74]). Note that for p to exist the lattice must be modular, and that we have:

$$\forall x,y \in L, \quad p(x \wedge y) \le \min(p(x), p(y)) \le \max(p(x), p(y)) \le p(x \vee y). \tag{1}$$

Now the relation defined by:

$$(P10) \quad \forall x,y \in L, \quad x \equiv y \leftrightarrow p(x \vee y)=p(x \wedge y)$$

is a *congruence* on L (Birkhoff [5, p. 77]) so that:

$$\forall x,y,z \in L, \quad x \equiv y \to (x \vee z)=(y \vee z), \qquad (x \wedge z)=(y \wedge z), \tag{2}$$

which in its turn implies:

$$\forall x,y,z \in L, \quad p(x \vee y)=p(x \wedge y) \to p(x \vee z)=p(y \vee z),$$

$$p(x \wedge z)=p(y \wedge z), \tag{3}$$

i.e. $x \equiv y$ means that y may be substituted freely for x in both L and p expressions without changing their value. Thus, with respect to the valuation p the relation $\equiv$ is one of *logical equivalence*.

Note that $x \equiv y$ is a relation on L, not an element in L. However, it is useful to consider the possibility of there existing within L an element z that *behaves* as $x \equiv y$ in that:

$$z = T \leftrightarrow x \equiv y. \tag{4}$$

We shall regard the symbol $x \equiv y$ as also standing for the class of such elements, if any exist. We can then write (4) as:

$$\forall x,y \in L, \quad p(x \equiv y) = 1 \leftrightarrow x \equiv y. \tag{5}$$

Note that use of the expression $p(x \equiv y)$ does not mean that all members of

the equivalence class have the same valuation. They need only have the same valuation when this is 1 and then the constraint expressed in (5) applies. This is a common feature of multivalued logic, i.e. that functions such as equivalence are constrained to have the obvious semantics when they express the truth of a relation but otherwise can take a variety of forms giving essentially different logics (Rescher [29]).

In terms of this structure some of the features of both probability and fuzzy logic are already apparent. P8 and P9 together ensure that the logic gives the standard truth tables for conjunction and disjunction when restricted to binary values. P9 is a property we associate with additivity of probability and inequality (1) brings in the max/min functions we associate with fuzzy logic, yet both features belong to the system defined so far.

2.2. Implication in a SUL as a distance function

We can complete this system by defining an implication function in terms of a natural distance on the lattice L. Define:

$$(\text{P11}) \quad \forall x, y \in L, \quad d(x, y) = p(x \wedge y) - p(x \vee y)$$

The function d is a quasi-metric on L (Birkhoff [5, p. 77]) satisfying:

$$\forall x \in L, \qquad d(x, x) = 0, \tag{6}$$

$$\forall x, y \in L, \qquad 0 \leq d(x, y) \leq 1, \tag{7}$$

$$\forall x, y, z \in L, \quad d(x, z) \leq d(x, y) + d(y, z). \tag{8}$$

The congruence of P10 is such that the quotient lattice of L under it has d as a true metric such that:

$$\forall x, y \in L, \quad d(x, y) = 0 \leftrightarrow x \equiv y. \tag{9}$$

Hence, consistent with the constraint of (5), we can define a measure of the equivalence between two elements in terms of the distance between them:

$$(\text{P12}) \quad \forall x, y \in L, \quad p(x \equiv y) = 1 - d(x, y) = 1 - p(x \vee y) + p(x \wedge y).$$

This gives all members of the equivalence class denoted by $x \equiv y$ the same valuation.

It is then natural to define a valuation of the extent to which x "implies" y by noting that if $x \supset y$ is true in standard logic, then $x \wedge y = x$ and $x \vee y = y$,

and letting the degree of equivalence of what would normally be equalities define:

$$\text{(P13)}\quad \forall x, y \in L,\quad p(x \supset y) = p(x \equiv x \wedge y) = 1 - d(x, x \wedge y)$$
$$= 1 - p(x) + p(x \wedge y) = 1 + p(y) - p(x \vee y)$$
$$= 1 - d(y, x \vee y) = p(x \equiv x \vee y).$$

Again, $x \supset y$ should be read as a symbol for a set of lattice elements, possibly empty, such that they satisfy the constraint of (P13).

Negation may be defined in the usual way (Prior [27, p. 50]) in terms of equivalence, and P5, P7 and P13 may be used to obtain a well-defined value for $p(\bar{x})$, the valuation of each member of the equivalence class $\bar{x}$, in terms of $p(x)$:

$$\text{(P14)}\quad \forall x \in L,\quad p(\bar{x}) = p(x \equiv F) = p(x \supset F) = 1 - p(x).$$

And we have conversely:

$$\forall x \in L,\quad p(x \equiv T) = p(T \supset x) = p(x). \tag{10}$$

The logical system we have defined through a valuation over a lattice of propositions is a simple and natural one that will be shown to include both probability and fuzzy logics as special cases. It has so many of the expected features and theorems of both probability and fuzzy logics that I have previously termed it a *standard uncertainty logic* (SUL, Gaines [12]). SUL defaults as expected in the binary case in that the definitions of valuations for conjunction disjunction, equivalence, implication and negation have the standard truth tables when the valuation can only take the values 0 and 1. It does not have the law of the excluded middle as a theorem. However, if elements exist within L that correspond to $x \equiv y$, $x \supset y$, and $\bar{x}$, then we have from P9 and P13 that:

$$\forall x, y \in L,\quad p(x \vee (x \supset y)) + p(x \wedge (x \supset y) = 1 + p(x \wedge y), \tag{11}$$

so that, substituting F for y:

$$\forall x \in L,\quad p(x \vee \bar{x}) + p(x \wedge \bar{x}) = 1, \tag{12}$$

showing that the *law of the excluded middle* ($p(x \vee \bar{x}) = 1$) and the *law of contradiction* ($p(x \wedge \bar{x}) = 0$) are equivalent in the logical system defined in that postulating one allows the other to be derived.

2.3. Some decision-theoretic results in a SUL

Certain practical results hold for a standard uncertainty logic without any further postulates that are key to a range of decision-theoretic applications. The implication valuation of P13 has the property that:

$$\forall x, y \in L, \quad p(y) = p(x \vee y) - 1 + p(x \supset y) \geq p(x) - (1 - p(x \supset y)), \quad (13)$$

which enables a lower bound to be placed on the truth value of y given those for x and $x \supset y$, thus allowing a multivalued form of *modus ponens*.

This result is particularly interesting if applied to the problem of paradoxical decisions resulting from long chains of inference. The classic paradox is that of deciding whether someone is bald: clearly someone with no hair is bald, and surely someone who differs from a bald man by having only one additional hair must also be considered bald, but then by repeated extrapolation someone with any amount of hair is bald. Classical logic has difficulty with this type of argument because it makes no provision for an inference to be almost, but not quite, valid: either $x \supset y$ is true or it is false. If we read x as $B(n-1)$: "a man with $n-1$ hairs is bald", and y as $B(n)$: "a man with n hairs is bald", then the paradox arises because $x \supset y$ is so nearly valid that we do not wish to deny it, yet when repeatedly applied it leads to unacceptable inferences, that is, from the truth of $B(0)$ we can infer the truth of $B(n)$ for any n.

SUL, however, allows for an inference rule to be nearly, but not quite valid, for example we may set:

$$p(x \supset y) = 1 - \alpha \quad \text{or} \quad p(B(n-1) \supset B(n)) = 1 - \alpha, \quad (14)$$

where α is very small, so that the truth value of $B(n-1) \supset B(n)$ is very near to, but not quite, 1. Then (13) gives us:

$$p(B(n)) \geq p(B(n-1)) - \alpha, \quad (15)$$

from which we may infer, by repeated application, that:

$$p(B(n)) \geq p(B(0)) - n \times \alpha, \quad (16)$$

so that, even though $B(0)$ might be true, and the inference rule $B(n-1) \supset B(n)$ is nearly true, the truth value of $B(n)$ declines with the length of the chain of inference necessary to derive it. This is an intuitively satisfying model for this type of paradox and its resolution.

When we are actually dealing with rules of inference that are valid in that $p(x \supset y) = 1$, we can derive from (13):

$$\forall x, y \in L, \quad p(x \supset y) = 1 \rightarrow p(y) \geq p(x), \quad (17)$$

so that the form of implication defined for a SUL satisfies the requirement (Lee and Chang [21]) that the assertion of $x \supset y$ may be used to infer that $p(y) \geq p(x)$. The result of (17) generalizes to a very significant inequality that, when y is constrained by rules of the form $x \supset y$ ("if x then y"), a lower bound of $\max(p(x))$ may be placed on $p(y)$:

$$\forall x, y \in L \text{ such that } x \supset y, \quad p(y) \geq \max(p(x)). \tag{18}$$

This is a key pattern of inference in the decision and control applications of fuzzy reasoning (Mamdani and Assilian [24]) where a number of "rules" are given that are then used to derive actions from actual data. It is worthwhile emphasizing that this result, although generally thought of as one of "fuzzy reasoning" actually applies to "probabilistic reasoning" also; that is, if we have a *rule* that from knowing x we may derive y, then if we know the probability that x has occurred we may infer that the probability that y has occurred is at least as great as this. Gaines [9] showed that the control strategy derived by Mamdani and Assilian [24] through the application of rules of fuzzy logic to their data was replicated when probabilistic rules were applied to the same data. Thus, eq. (18) expresses a common pattern of inference in both probabilisitic and fuzzy decision making.

2.4. Derivation of fuzzy and probability logics from a SUL

The unusual feature of SUL is that it is *non-truth-functional* in that the values of $p(x \vee y)$, $p(x \wedge y)$, $p(x \equiv y)$ and $p(x \supset y)$ cannot be obtained from $p(x)$ and $p(y)$. However, there is only one degree of freedom since P9 ensures that the fixing of $p(x \wedge y)$ also fixes $p(x \vee y)$, and vice versa, and even this freedom is restricted by the inequality of (1). If we require the SUL to be strongly truth-functional such that the valuation of any connective can be determined in terms of the valuations of its constituents, then the arguments of Bellman and Giertz [3] show that the outer inequalities of (1) become equalities and the logic is Lukasiewicz's infinitely valued logic (Rescher [29, section 6]). This has the connectives used by Zadeh [36] for fuzzy logic;

(L1) $0 \leq p(x) \leq 1$,

(L2) $p(\bar{x}) = 1 - p(x)$,

(L3) $p(x \wedge y) = \min(p(x), p(y))$,

(L4) $p(x \vee y) = \max(p(x), p(y))$,

(L5) $p(x \supset y) = \min(1, 1 - p(x) + p(y))$,

(L6) $p(x \equiv y) = \min(1 - p(x) + p(y), 1 + p(x) - p(y))$.

This is a significant result because it explains some of the attraction of fuzzy logic. The requirement of truth functionality is one which makes the logic computationally tractable, both for people and for computers. Whether the logic is actually truth-functional in a particular application clearly depends on the semantics of that application, but the assumption that leads to the connectives of fuzzy logic is a way of resolving the uncertainty about derived truth values in a consistent fashion. Any other resolution leaves the valuation of a compound proposition dependent not just upon the valuation of its constituents but also upon their structure. *With any other assumption we have to remember not only the truth values of derived propositions but also the way in which they were derived.*

This truth-functionality does not hold for probability logic and this may be derived from SUL by an alternative additional assumption, namely that the law of contradiction holds so that $p(x \vee \bar{x}) = 1$. Eq. (11) shows that this is equivalent to assuming that the law of the excluded middle holds, and either assumption gives a standard probability logic (Rescher [29, p. 185]):

(R1) $0 \leq p(x)$,

(R2) $p(x \vee y) = p(x) + p(y)$ if x and y are mutually exclusive,

(R3) $p(x) = p(y)$ if x and y are logically equivalent,

(R4) $p(x \vee \bar{x}) = 1$,

which is also the system defined by Fenstad [8] to represent probabilities on a first-order language.

Thus, a SUL system of valuations on a lattice as defined above subsumes both probability and fuzzy logics and enables key inference rules common to both to be derived. The logics only diverge in a final step whereby the use of fuzzy logic assumes strong functionality and the use of probability logic assumes the law of the excluded middle. I emphasize this not to suggest that the difference between the two logics is trivial, but rather to show that it does not necessarily lie in major part in the connectives used. For many applications a SUL is adequate and this encompasses both forms of connective. It is rather to the forms of the applications themselves that we should look for the interesting differences between the developments in decision analysis covered in this book and the more standard approaches.

2.5. *Axiomatic form of SUL*

There is an alternative approach to the SUL and its relation to fuzzy and probability logics that brings out further features of the relationship. In particular it shows how both logics relate to the analysis of *possibilistic* systems

and decision-making (Gaines and Kohout [15], Zadeh [38]. For this approach we go back to the formal axiomatic systems underlying the logics.

Lukasiewicz infinitely-valued logic has the underlying formal system (Rescher [29], Gaines [10]) of the following four axioms in terms of falsity and implication as primitives:

(A1) $\forall x, \quad F \supset x$

that from the assertion of falsity, or a contradiction, any proposition may be inferred;

(A2) $\forall x, y \quad x \supset (y \supset x)$

that a true proposition may be inferred from any proposition, the so-called paradox of material implication;

(A3) $\forall x, y \quad ((x \supset y) \supset y)) \supset ((y \supset x) \supset x)$

that disjunction and conjunction are symmetric (see DD and DC following);

(A4) $\forall x, y, z \quad (x \supset y) \supset ((y \supset z) \supset (x \supset z))$

that implication is transitive.

To these must be added the inference rules of substitution, i.e. that any well-formed formula may be substituted uniformly for any variable in a theorem, and modus ponens:

(MP) $\forall x, y, \quad x, x \supset y \rightarrow y$

that from x and $x \supset y$ we may infer y.

The remaining logical connectives may then be defined as:

(DN) Negation	$\bar{x}$	for	$x \supset F$,
(DD) Disjunction	$x \vee y$	for	$(x \supset y) \supset y$,
(DC) Conjunction	$x \wedge y$	for	$\sim (\sim x \vee \sim y)$,
(DE) Equivalence	$x \equiv y$	for	$(x \supset y) \wedge (y \supset x)$.

This logical system reduces to the standard propositional calculus (PC) if the additional axiom is assumed that:

(A5) $\forall x, y, \quad ((x \supset y) \supset x) \supset x$

which, from DC, is equivalent to:

(A5′) $\forall x, y, \quad (x \supset y) \vee x$

which, if we substitute F for y, gives us:

$$\forall x, \quad (x \supset F) \vee x \quad \text{or} \quad \bar{x} \vee x,$$

which is the law of the excluded middle.

The axioms A1 through A4, inference rules, and definitions are satisfied by a SUL and serve to define it in non-numeric terms. They are generally thought of as axioms for Lukasiewicz logic, and Wajsberg [33] derives the numeric form given in L1 through L6 from them. However, he assumes that the logic is truth-functional and hence his result is parallel to that of the previous section – that a SUL that is truth-functional is precisely Lukasiewicz's infinitely-valued logic.

The other parallel result is that a probability logic as defined by R1 through R4 also obeys these axioms together with A5 also – that a SUL with the law of the excluded middle is precisely Rescher's probability logic. Rescher [28] has shown that probability logic is formally equivalent to the modal logic S5, and this is apparent in axioms A1 through A5 being those of the implicational fragment of S5 provided $\supset$ is regarded as a "strict" implication (Lemmon, Meredith, Prior Thomas [22], Anderson Belnap [2]).

It is interesting to summarize these results by relating them back to some of the proposed applications of the logical calculi also. Gaines and Kohout [15]

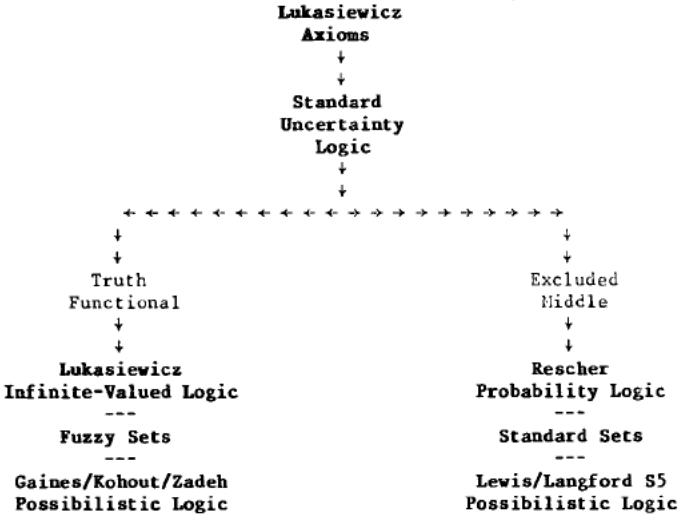

Fig. 1. Relations between SUL and possibilistic logics.

and Zadeh [38] have proposed that Lukasiewicz or fuzzy logic be regarded as a basis for a *possibilistic* system theory and have given illustrations of its applications to database systems (Gaines [14]) and to the extension of probabilistic sequential system identification (Gaines [11]) to uncertain sequential data (Gaines [13]). The modal logic S5 is the classical logic of possibility of Lewis and Langford [23], and is also the system that Rescher [28] has shown to underly probability logic which is itself used in practice as a basis for possibilistic system theory. The standard uncertainty logic developed in this section draws all these systems together and shows both their close interrelationships and also their essential differences; fig. 1 illustrates this in diagrammatic form.

2.6. The origin of numeric truth values

One feature of both fuzzy and probabilistic logics, as models of uncertain reasoning, that is often taken as counter-intuitive is their use of numbers to represent probabilities or degrees of membership, and their use of exact logical formulae for deriving results. The analysis of the bald man paradox above shows how a SUL, and hence probability and fuzzy logics, can adequately represent approximate or uncertain reasoning. From the underlying axiomatic structure of these logics it is also possible to show how the numeric truth values arise in a simple and natural way.

It is convenient to state this in terms of the resolution of a further paradox of classical logic, i.e. that of Russell's barber who shaves everyone in a village who do not shave themselves – does he shave himself? This corresponds to the fault in Frege's axiom of comprehension in naive set theory that Russell discovered – that not all predicates define sets (Kneale and Kneale [19, ch. 11]). It led to Russell's theory of types (Copi [7]) as a solution to the problem that retained the structure of classical logic without leading to inconsistencies in set theory by placing restrictions on those predicates which could define sets. It has long been suggested that an alternative approach to the problem is to change the rules of classical logic, but this has proved difficult. Maydole [26] showed that variants of Russell's paradox could be generated for virtually all variant logics except two attributed to Post and Lukasiewicz. These two cases were left open, as Maydole's technique did not generate paradoxes, but it was not possible to show using his methodology that some other technique might not do so. White [34] showed that the quantified form of Lukasiewicz infinitely valued logic did lead to a consistent set theory having a completely unrestricted axiom of comprehension – any predicate defines a set.

Analysing Russell's barber paradox illustrates the way in which *new* truth values are generated in a SUL. If we take S to mean that the barber shaves himself, then the following two lines of reasoning apply: if S is true then the

barber shaves himself so that he is not shaved by the barber and hence S is false, i.e.

$$S \supset \bar{S}; \tag{19}$$

however, if S is false, then the barber does not shave himself so that he is shaved by the barber and hence S is true, i.e.

$$\bar{S} \supset S. \tag{20}$$

These taken together give us that:

$$S \equiv \bar{S}, \tag{21}$$

which means that S is an impossible proposition for classical logic – it is both true and false. However, having already noted that the law of contradiction does not hold in a SUL, we should not be surprised to find that S is a perfectly legitimate proposition in that logic. Indeed from (21) and P14 we may derive that:

$$p(S) = p(\bar{S}) = 1 - p(S), \tag{22}$$

so that:

$$p(S) = 1/2. \tag{23}$$

Now this should be a surprising result because nowhere in the definition of a SUL have we defined the number $1/2$. It has arisen from the rules given on the basis that if there is a proposition S with a truth value that satisfies the predicates defined by Russell and the rules of an SUL then its truth value must be $1/2$. Thus, the *assumption of existence* of such a proposition has generated the need for, and effectively *brought into being*, the intermediate truth value, $1/2$. Varela [32] has used this same line of reasoning to bring a third truth value into Brown's [6] logic of distinctions in order to model the phenomena of self-reference.

The technique of using what are paradoxes of classical logic to generate truth values in a multivalued logic can be extended to give a generator for any required value. Consider an expression of the form of (20), $\bar{x} \supset x$, then from P13 and P14:

$$p(\bar{x} \supset x) = 1 - p(\bar{x}) + p(x \wedge \bar{x}) = p(x) + p(x \wedge \bar{x}). \tag{24}$$

However, from (1):

$$p(x \wedge \bar{x}) \leqslant \min(p(x), p(\bar{x})) = \min(p(x), 1 - p(x)), \tag{25}$$

so that:

$$p(x) \leq 1/2 \rightarrow p(x \wedge \bar{x}) = p(x), \tag{26}$$

and hence:

$$p(x) \leq 1/2 \rightarrow p(x) = p(\bar{x} \supset x)/2. \tag{27}$$

Now consider a series of propositions, $S(i)$, that are defined recursively by:

$$(\bar{S}(i) \supset S(i)) \equiv S(i-1), \tag{28}$$

with

$$S(0) \equiv F \supset F \quad \text{and} \quad S(1) \equiv S, \tag{29}$$

where S is the proposition defined by (21). Then we have from (27) that:

$$p(B(i)) = (1/2)^i, \tag{30}$$

so that the assumption of the existence of the $S(i)$ generates an infinite sequence of propositional constants with truth values having binary fractional powers, in place value notation, of the form 0.00001000000... in radix 2 for some position of one in a sequence of zeros.

This may be extended to the generation of an arbitrary truth value by considering an expression of the form:

$$R = (\bar{S}(i) \supset (\bar{S}(j) \supset (\bar{S}(k) \supset (\ldots \supset S(n))))), \tag{31}$$

where $i < j < k < \ldots < n$. Then:

$$p(R) = (1/2)^i + (1/2)^j + (1/2)^k + \ldots + (1/2)^n. \tag{32}$$

Hence, we may use the propositional sequence $S(i)$ to generate any truth value by expressing it as a fractional binary expansion and putting the proposition $S(i)$ in an expression of the form of (31) if there is a 1 in the ith binary place of the expansion. That this derivation applies to a SUL in general, and hence to both fuzzy and probabilistic logics, again emphasizes the link between them since the modelling of the modal logic S5 through infinite binary sequences is a well-known result in temporal logic (Rescher [29, p. 194]).

This derivation of numeric values from the basic axioms of the system, which themselves introduce only the numbers 0 and 1, is a significant indicator of the source of numeric truth values in uncertain reasoning. They may be seen as arising out of an *order* relation on propositions, which may be one of

probability, credibility, plausibility or some such term dependent on how we view the situation creating uncertainty. Under uncertainty we may not be able to make clear-cut assignments of truth or falsity to statements. However, logical constraints may lead us to assertions of the form that "even though neither x nor y is definitely true or false, x is more probable (credible, plausible) than y". This relation can be expressed by assigning x some non-zero truth value and y some greater, non-unity truth value. Now, if a further statement is made that is more probable (credible, plausible) than x and less than y, then it needs to be placed between them. Thus, in terms of our fractional binary expansions, longer and longer sequences may become necessary to cope with the distinctions to be made in ordering the statements.

The use of numbers implies that we regard the uncertainties about statements as being well-ordered. However, the arguments of this section apply also to the partially ordered algebraic structures that Goguen [18] introduces as generalized truth-sets. We need some mechanism of interpolation if we are to encode the differentiations we wish to make in our statements about the world and that is how the numbers arise from the logical structure of our statements.

As a final example of the way in which some of the stereotypes distinctions between fuzzy and probabilistic logics have no basic foundation but arise out of use, consider the arithmetic mean, or averaging operator, over a set of truth values. This arises naturally in statistical operations involving probabilities but seems unnatural in fuzzy logic where a min or max operation is a more "natural" way of combining numbers. However, the results of (24) through (27) can be extended to give a natural averaging operating in a SUL and hence in both fuzzy and probability logics. Suppose we have n propositions $X(i)$, $1 \leq i \leq n$, then we can first use an extended version of (24), analogous to (31), to implicitly define related propositions $Y(i)$:

$$X(i) = (\overline{Y}(i) \supset (\overline{Y}(i) \supset (\overline{Y}(\ldots)))), \tag{33}$$

where there are $nY(i)$ on the right-hand side, so that:

$$p(Y(i)) = p(X(i))/n. \tag{34}$$

Then we use the $Y(i)$ in an expression, again analogous to (31), to define a proposition Z:

$$Z = (\overline{Y}(1) \supset (\overline{Y}(2) \supset (\overline{Y}(3) \supset (\ldots \supset Y(n))))), \tag{35}$$

so that:

$$\begin{aligned} p(Z) &= p(Y(1)) + p(Y(2)) + p(Y(3)) + \ldots + p(Y(n)) \\ &= (p(X(1)) + p(X(2)) + p(X(3)) + \ldots + p(X(n)))/n. \end{aligned} \tag{36}$$

Thus, the arithmetic mean of a set of truth values may be constructed within a SUL and hence is a legitimate operation of both fuzzy and probability logics.

3. Conclusions

One major conclusion to be drawn from this paper should now be obvious – that fuzzy set theory and probability theory should be viewed not as rivals, and not necessarily even as complementary, but rather as similar logical systems, having a common core that is adequate for many aspects of decision analysis, and differing in certain well-defined features that may, or may not, be relevant in particular applications.

A standard uncertainty logic (SUL) based on a system of valuations on a lattice subsumes both probability and fuzzy logics and enables key inference rules common to both to be derived. The logics only diverge in a final step whereby the use of fuzzy logic assumes strong truth functionality and the use of probability logic assumes the law of the excluded middle. This is not to suggest that the difference between the two logics is trivial, but rather to show that it does not necessarily lie in major part in the connectives used. For many applications a SUL is adequate and this encompasses both forms of connectives.

In the context of this book we can see that fuzzy logic, by not requiring the laws of contradiction and the excluded middle to hold, allows for borderline cases – for predicates that do not necessarily either exclude or include particular cases. The gradation of degrees of membership allows an order relation to be placed on set inclusion and is the basis for dealing with varying degrees of uncertainty in decision. It is this aspect of probability that we use, and have found so effective, in many practical applications – and we intuitively do not make use of those parts of probability logic that have no relevance to the application or would lead to meaningless, or incorrect, results if applied. In doing this we are essentially working with the weaker structure on a SUL without making this explicit.

The assumption of truth-functionality which takes us from a SUL to fuzzy logic as normally defined may be seen as a computational necessity in many applications – it allows us to forget the structure of propositions and remember only their truth values. Whether or not we can get away with these particular applications is a function of those applications and the robustness of the analysis being used. For real-world applications with uncertain inference rules as well as uncertain data the validity of refined arguments deteriorates rapidly and we have to use short chains of reasoning – this is one plausible argument for the success of fuzzy reasoning despite the assumptions made.

This paper has also indicated that there are strong arguments in mathematical set theory and logic for regarding the Lukasiewicz infinitely valued logic in

its axiomatic form as a key foundation for abstract reasoning. It will be a long and laborious process to re-develop the foundations of mathematics, logic, and ultimately decision analysis using a non-standard logic. The benefits of even the preliminary steps in this exercise, however, are to focus upon those parts of the theoretical systems which we use that are key to our applications, in contrast to those which are superstitious carry-overs doing no harm, and those parts which, as engineers, we quietly discard when it comes to the real world.

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