Department of Electrical Engineering Science

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DISCRETE SYSTEMS AND FUZZY REASONING

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Digitization of *Discrete Systems and Fuzzy Reasoning*

There is growing historical interest in the early days of fuzzy linguistic control. This document is a digitized version of the proceedings of the first workshop that Abe Mamdani and I organized in 1976, the year after I published his and Assilian's paper on Assilian's doctoral research in the International Journal of Man-Machine Studies. It contains some 8 papers on fuzzy control arising out of that research as it began to attract industrial interest, together with related papers on fuzzy reasoning and multi-valued logic that reflect the state-of-the-art in 1976.

I printed a substantial number of the original proceedings at the University of Essex and distributed them widely at the time, but they are difficult to obtain now, and making them available in digital form seemed a useful task.

I dedicate this version of those proceedings from some 36 years ago to two close friends and colleagues from that time who are no longer with us, Abe Mamdani and Ladislav Kohout. We worked hard together, shared many exciting ideas and ideals, had a lot of fun, and enjoyed living in interesting times.

Brian Gaines, Vancouver Island, April, 2012

EES-MMS-DSFR-76

DISCRETE SYSTEMS AND FUZZY REASONING

Edited by:-

E. H. Mamdani & B.R. Gaines

Proceedings of a Workshop at

Queen Mary College, held on

Friday 9 January '76.

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FOREWORD

The Fuzzy Workshop held at Queen Mary College on Friday 9th January 1976 was aimed at bringing together the several groups working on fuzzy reasoning in the U.K. It echoes the similar workshops held in the U.S.A., Japan and Europe, in attempting to consolidate and cross-fertilize this rapidly growing field.

The Workshop itself was a successful occasion for the participants. These proceedings make some of the material presented available in written form to the participants, and to a wider audience. Note that these are both working papers and papers published elsewhere in these proceedings (an indication is given if the proceedings are a proper reference). Also there is no direct link between the presented papers and the written ones. The Workshop was an exchange of views - the proceedings are an exchange of information.

As organizers we owe a vote of thanks to the several postgraduate students of Queen Mary College who helped to make the day run smoothly, to the catering staff who excelled themselves, and finally to the participants who made it worthwhile.

E. H. Mamdani

B. R. Gaines

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WORKSHOP ON DISCRETE SYSTEMS

AND FUZZY REASONING.

Friday 9th January 1976

Queen Mary College, Department of

Electrical and Electronic Engineering

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VENUE:- Lecture Room 102 in Department of Chemistry, Q. M. C.

FINAL PROGRAMME

9.30 COFFEE AND REGISTRATION

10.00 WELCOME, by E. H. Mamdani

10.05 KEYNOTE LECTURE
"Understanding Uncertainty" B.R. GAINES

SESSION I - Chairman, B.R. Gaines

10.45 "Fuzzy Integrals" M. SUGENO

11.00 "Aspects of Multivalued Logic" V. PINKAVA

11.15 "Models of Decision Making in the Allocation of Resources" P.K. M'PHERSON

11.30	"Ordinary Language Algorithms and Coping with Fuzziness in Everyday Life"	B. LEWIS
11.45	"Discovery of Generative Structure in Empirical Data"	G. KLIR
12.06	Discussion	
12.30	LUNCH	

SESSION II - Chairman, E.H. Mamdani

13.30	"Control Application of Fuzzy Logic"	P. KING
14.00	"Implementation and Evaluation of a Fuzzy Control Algorithm for a Sinter Plant".	M. BRAAE, G.P. RAO & M.T. HAGUE
14.20	"Analysis of a Fuzzy Controller"	W.KICKERT & N.BAAKLINI
14.35	"Multivalued Logics of Movement Hierarchies"	L.J. KOHOUT
14.50	"Linguistic Reasoning in AL"	P.J. HAYES
15.10	Discussion	
15.45	TEA	

SESSION III - Chairman, P.K. M'Pherson

16.00	"Survey of Learning Automata"	I.H. WITTEN
16.30	"Semantic Memories"	J.M.V. SLACK
16.45	"Discrete Pattern Transformation"	D.G. TONGE
17.00	"How Natural are the Formulation of Fuzzy Set Theory?"	J.L. CAMPBELL
17.15	"Comments on Fuzzy Relation in Representing Cognitive Structures"	G. PASK
17.35	Discussion	
18.00	FREE PERIOD - Drinks available at Bar.	
18.30	WORKSHOP DINNER	

Organisers: B.R. Gaines and E.H. Mamdani

WORKSHOP ON DISCRETE SYSTEMS AND FUZZY REASONING

Friday 9th January, 1976

Q.M.C. London

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WORKSHOP ON DISCRETE SYSTEMS AND FUZZY REASONING

Friday 9th January, 1976

Discussion Session I & II reported by T.J Procyk

Chaired by Dr. E.H. Mamdani

Professor M'Pherson began the discussion by raising the question of the relative cost-effectiveness of a fuzzy,two-term and human controller for the sinter plant. Mr. Hague said that the fuzzy controller was more costly than a two-term one even if implemented in table look-up form on a microprocessor. He was in agreement with Dr. Mamdani that using a microprocessor is often a false economy owing to the time required to interface it and program it. Dr. Mamdani was of the opinion that full facilities of a microprocessor are not required to implement a fuzzy controller but just a table look-up memory.

The most fascinating piece of work coming from the session, according to Professor Gaines, was the move upwards into implementing a second hierarchical level capable of supervising the lower one. The ability to define a fuzzy goal from which the control rules evolve was a significant step forward. This 'bottom up' approach of starting with a simple problem and building it up in complexity was a philosophy that Dr. Mamdani said he adopted and suggested it as a useful approach for tackling complex problems. Professor M'Pherson, who thought that Fuzzy Logic theory had not yet developed sufficiently to be applicable to many classes of problems, agreed with Dr. Mamdani that Zadeh's theory is only 15 years old and still in its youth. Dr. Mamdani believed that Zadeh is constantly revising his theory and extending it so it should not be regarded as final. The added merit of Fuzzy Logic theory was that it is still open to modification unlike other multi-valued logics.

Mr. Hague wished to point out that because of the difficulty of designing non-linear controllers the success of the fuzzy controller is an important step in controller design. The results obtained by Professor Gaines from analysing the steam engine controller showed that it was very robust and insensitive to small changes in its structure. The reason why many classical controllers were sensitive was, according to him, because of the stress put on exactness in calculations which produced numerical differences smaller than the noise margin. Dr. Mamdani added that the source of this robustness lay in the fact that the fuzzy controller was not a pedantic one but, from common sense, a reasonable one which consequently lent itself to a wide range

Cont'd.....

of successful applications.

The concern for a lack of suitable stability theory for fuzzy controllers and their trial and error synthesis was voiced by Professor Hammond. Dr. Mamdani replied that stability theory is always used with a mathematical model anyway which is not always exact. Consequently full confidence in a control system's stability is never justified. He did not think that such a theory existed since there is no equivalent of a frequency domain, in which stability is tested, for discrete systems. He added that because of the common sense nature of the rules runaway instability cannot arise. The oscillatory type of instability is less serious and can be cured by tuning the rules.

Professor Gaines put forward a state-space approach which could be used for stability analysis. It consisted of the state-space marked with areas over which individual rules have control and from this the system trajectory could be determined. The main objection to this method was made by Mr. Kickert saying that it was only practical for two dimensions while instability was only important for 3rd order systems upwards.

Professor M'Pherson gave examples of some fast and accurate systems found in the defence industry for which fuzzy logic could not be conceivably used. This was very true, according to Dr. Mamdani, who did not regard fuzzy logic as an alternative answer to control problems but as useful in certain applications for example the sinter plant or cement kiln. Professor M'Pherson nevertheless admitted that fuzzy logic should at least be tried in new applications and not dismissed at the outset.

Dr. Smith stated that he was not concerned with stability as much as the control engineers were because he did not believe that systems like management information or air traffic control, with which he is concerned, would ever become closed-loop. In his opinion fuzzy logic had a place in such systems as a heuristic aid or guide with a human being present in the loop. Stability analysis was then not of such vital concern.

On a historical note Professor Gaines, concluded the discussion by commenting on the tremendous disruption caused by the 2nd World War and the ensuing period to work in multi-valued logics. Only now has the work which terminated at the outbreak of the war been again revived and started to gain interest.

FUZZY CONTROL OF RAW MIX PERMEABILITY AT A SINTER PLANT

G.A. Carter and M.J. Hague

British Steel Corporation,
General Steels Division,
Research Organisation (Teesside).

INTRODUCTION

The use of fuzzy set theory to design controllers has been pursued as a combined project by the University of Manchester Institute of Science and Technology and the British Steel Corporation. The project has a two-fold purpose:

(i) To establish the worth of a "fuzzy set" theory and fuzzy logic to practical plant control.

(ii) To provide a generalised control tool for systems where the plant dynamics are poorly described.

Control system development requires some basic knowledge of the dynamics and statics of the plant to be controlled. The knowledge is conventionally obtained from one of two sources:

(i) A theoretical relationship between the controller variables and the controlled variables.

(ii) A statistical appraisal of input and output signal spectral densities.

The ability of using fuzzy set theory to "describe" a controller to a computer opens up the possibility of utilising performance criteria used by good operators to calculate an advised control action to all operators. The application to blast furnace control, where computer models are rarely as accurate as good operators, is an obvious future consideration for fuzzy control advice.

At a Cleveland sinter plant, a two term controller exists for the control of water flow to a mixing drum and the opportunity has been taken to try fuzzy logic control so that some comparisons can be made. The Cleveland scheme represents a good test bed since the plant is non-linear and there exists large measurement noise and input disturbances.

THE PLANT

(a) Operation

Raw mix of iron ore (60%), sinter fines (35%), coke (4%) and flux addition (1%) are fed to a mixing drum where water is added to optimise the permeability of the mixture just prior to laying it on the grate for sintering. If the mix is too dry, the bed will choke and little air can be drawn through the bed for ignition and combustion. Thus the time for the bed to "burn through" will be long and the production rate low. Similarly, the mix must not be too wet otherwise the bed will sludge and again production rate will be low. This is described diagramatically for the sinter plant by Figure 1. The shape of the permeability/moisture curve is not fixed since it will depend on granulometry and the proportioning of the materials, both of which have to be changed for operational reasons.

Manual control of water addition which is the per shift/per hour selection of best water valve position, allows considerable fluctuation of permeability and therefore sintering rate to occur. The automatic control of water addition measures the pre-mix permeability and adjusts the water to achieve a set point value. The set point value is near to but on the dry side of maximum.

(b) Plant Dynamics

The control scheme at Cleveland Sinter Plant is shown in outline by Figure 2. The transfer function between water addition and permeability to a first approximation is of the form -

$$\frac{\text{Change in permeability}}{\text{Change in water}}(s) = G_o \frac{e^{-sT_1}}{1 + sT_2} \qquad \ldots (1)$$

T_1 = 20 seconds

T_2 = 30 seconds

The permeability is "sampled" once every 30 seconds. Uncorrelated measurement noise is assessed at between 10% and 30% of the total measured signal. The gain term 'Go' is the slope of the permeability/moisture curve.

SIMULATION STUDIES

A simulation of the performance of the sinter plant and the fuzzy algorithm was used as a dynamic check on the viability of the proposed controller.

The plant was represented by a transfer function of the form of equation (1) with the gain term G_0 represented by an adjustable non-linear function to simulate the non-linear relationship between moisture and permeability. In this way a range of characteristics similar to those shown in Figure 1 were simulated. The control rules relate error and sum of error to changes in water addition as shown in Figure 3.

Initial tuning of the controller was done by adjusting the scale factors associated with the look-up table using a procedure similar to that used when tuning a conventional controller.

The overall performance of the system was assessed in a number of ways:

(a) It was found that the transient performance was only slightly affected by changing the time constant in the range 2:1 and changing the dead time in the range 1.5 to 1. Performance was also satisfactory for process gain changes in the range 4:1.

(b) Changing the permeability moisture relationship to simulate changes to the raw material characteristics and to the initial moisture content did not result in any significant degredation in performance, provided the plant did not move into the region of negative slope.

(c) The response of the system to step changes in permeability set point was compared with that obtained with a conventional PI controller. A range of gains were used and similar results obtained, except that when the onset of instability was reached the fuzzy controller gave a more gradual run away.

(d) The effect of measurement noise on the system was compared with that obtained when a PI controller was used. It was found that the fuzzy controller was better than the PI controller in that it gave a further reduction of between 10% and 15% in the mean squared error when the simulated noise levels were similar to those expected on the plant.

The results indicated that the fuzzy controller was sufficiently insensitive to plant characteristics to justify a plant trial and that it should operate satisfactorily on signals containing noise.

RESULTS OF PLANT TRIAL

A trial was carried out at Cleveland Sinter Plant, Teesside, on No.4 Strand during September and October, 1975. The two-term controller was replaced by a portable PLP 11/10 mini computer (8 k store).

The system was allowed to "track" for a period of about an hour to establish the "manual" level of permeability variations. Then the computer was switched to "control" and, after transients had decayed (16 minutes), the reduction in permeability deviation, manual to fuzzy logic control, was noted. A reduction in SD of 40% was recorded (Table 1). A similar trial, in two-term control, records a 33% reduction (Table 1) which is typical. The range of reduction achieved with two-term control varies considerably from 20% to 40%. Typical differential pressure (which varies inversely with permeability) versus time recordings are given in Figure 4.

The trial was repeated on a later occasion when fuzzy control again, without any on-plant tuning, achieved a 23% SD reduction. Some attempt was made at 'on line' tuning. The scale of the control fuzzy set was reduced by half; the result was a 38% reduction in permeability standard deviation compared with manual. Increasing the scale of the set by two indicated that control was ineffective with a marginal increase in permeability variation.

DISCUSSION OF PLANT RESULTS

The essential part of this experiment was to demonstrate that a new control theory, using fuzzy logic could be used to control real plant containing noise and non-linearity. The results amply demonstrate that fuzzy set control meets the requirements on this plant.

FURTHER WORK

Confidence has been obtained in fuzzy logic control and it is now intended to apply the method to plant where the require-

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ment for little or no on plant tuning is matched by a poor knowledge of the plant, yet whose control philosophy can be written down as a set of linguistics.

ACKNOWLEDGEMENTS

The authors wish to acknowledge the assistance given to this project by Mr. C.G. Bloore and Mr. G. Summers who respectively carried out the simulation work and plant trials as part of their masters dissertations. Also the work of Mr. K. Crudgington, Research Organisation of the British Steel Corporation for providing the technical expertise on plant and the plant management for allowing the trials to take place. Acknowledgement is also made to the British Scientific Research Council for provision of computing facilities for simulation studies.

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1. C.G. Bloore: "An Heuristic Adaptive Controller for a Sinter Plant". M.Sc. Dissertation, University of Manchester, November 1974.

2. G. Summers: "Digital Control of a Sinter Plant", M.Sc. Dissertation, University of Manchester, December 1975.

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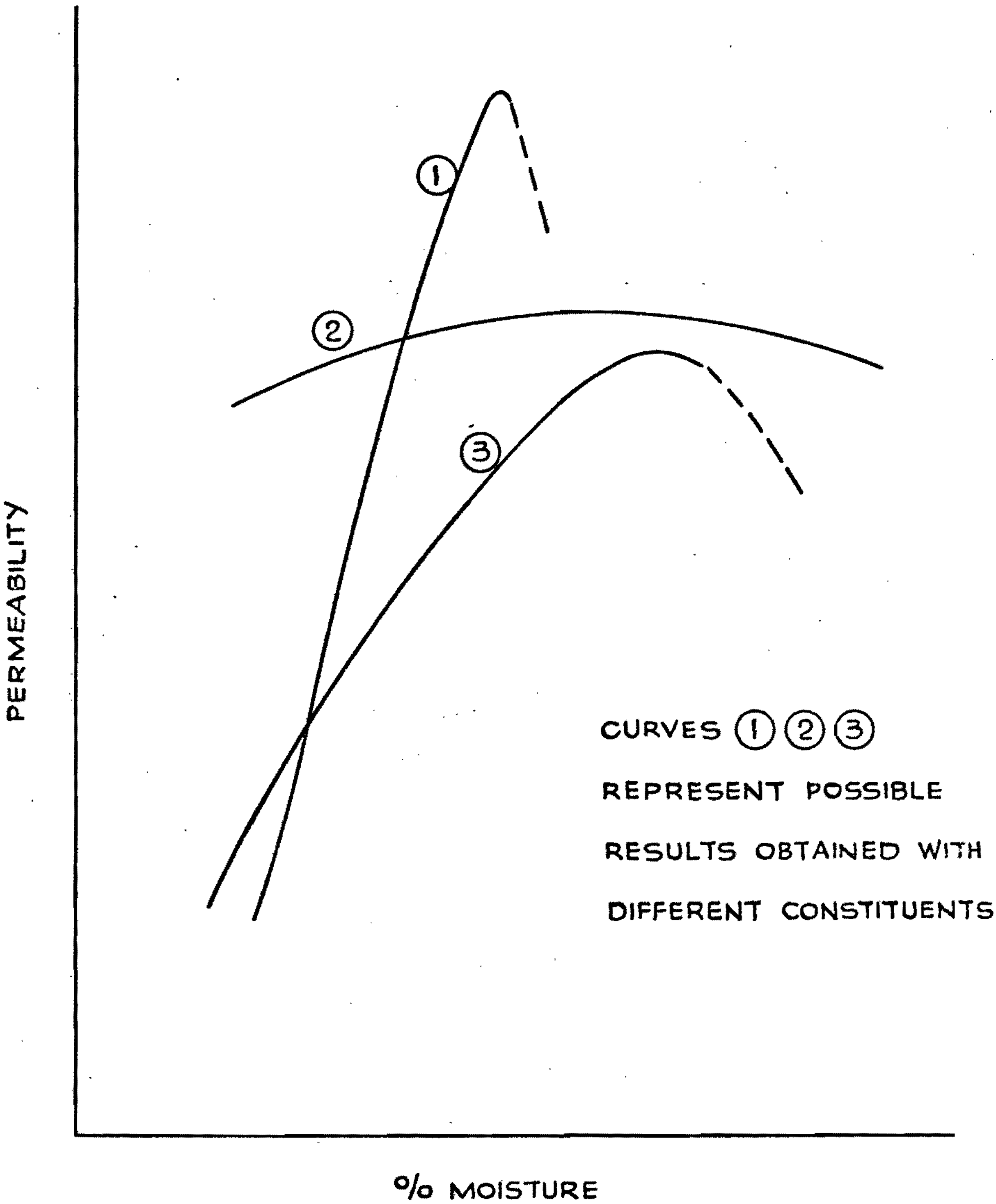
19th January, 1976.



TYPICAL RELATIONSHIP BETWEEN MOISTURE AND PERMEABILITY IN A RAW MIX SAMPLE

FIG. 1.

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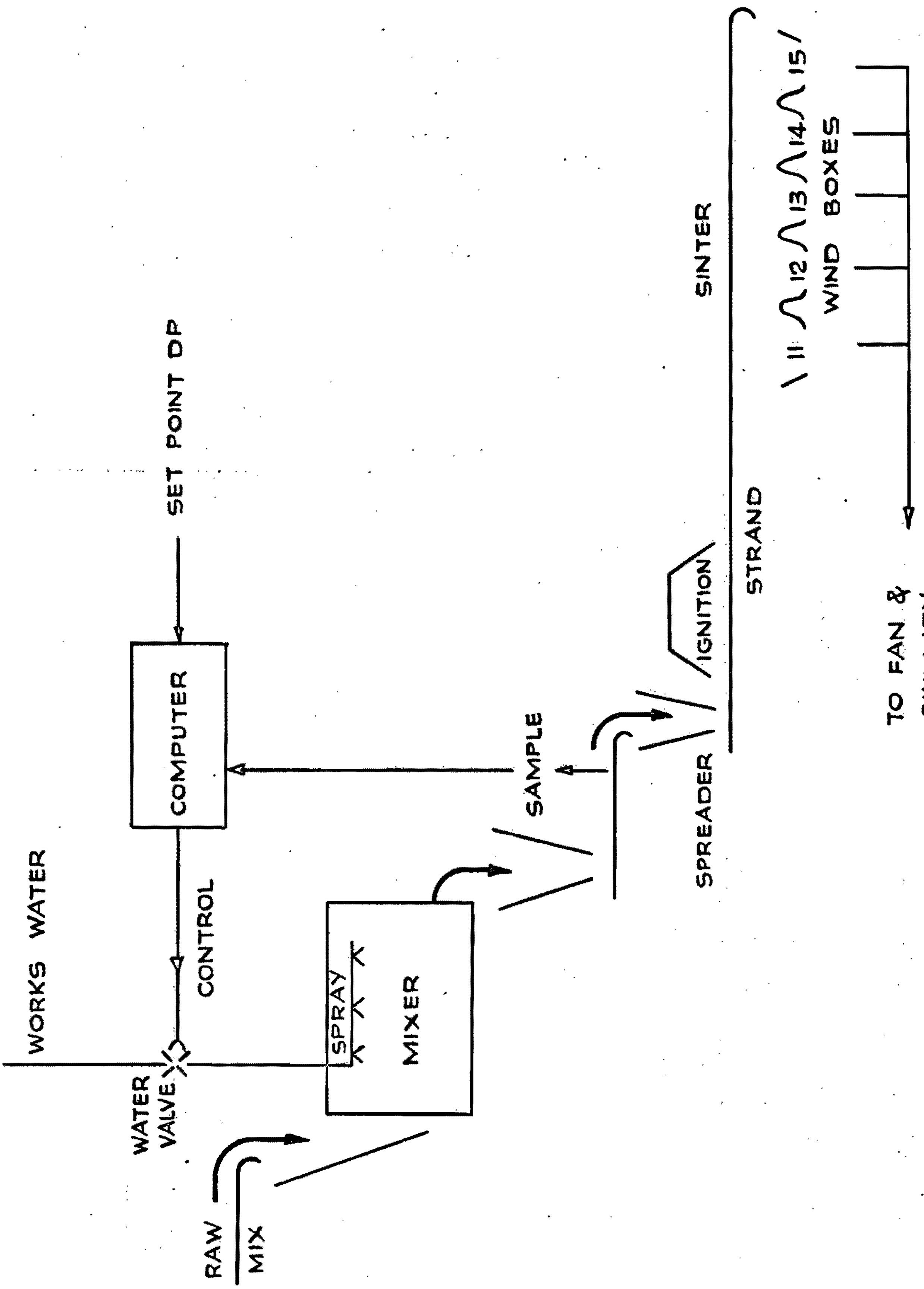


A DIAGRAMMATIC REPRESENTATION OF THE PERMEABILITY CONTROL SCHEME

FIG. 2.

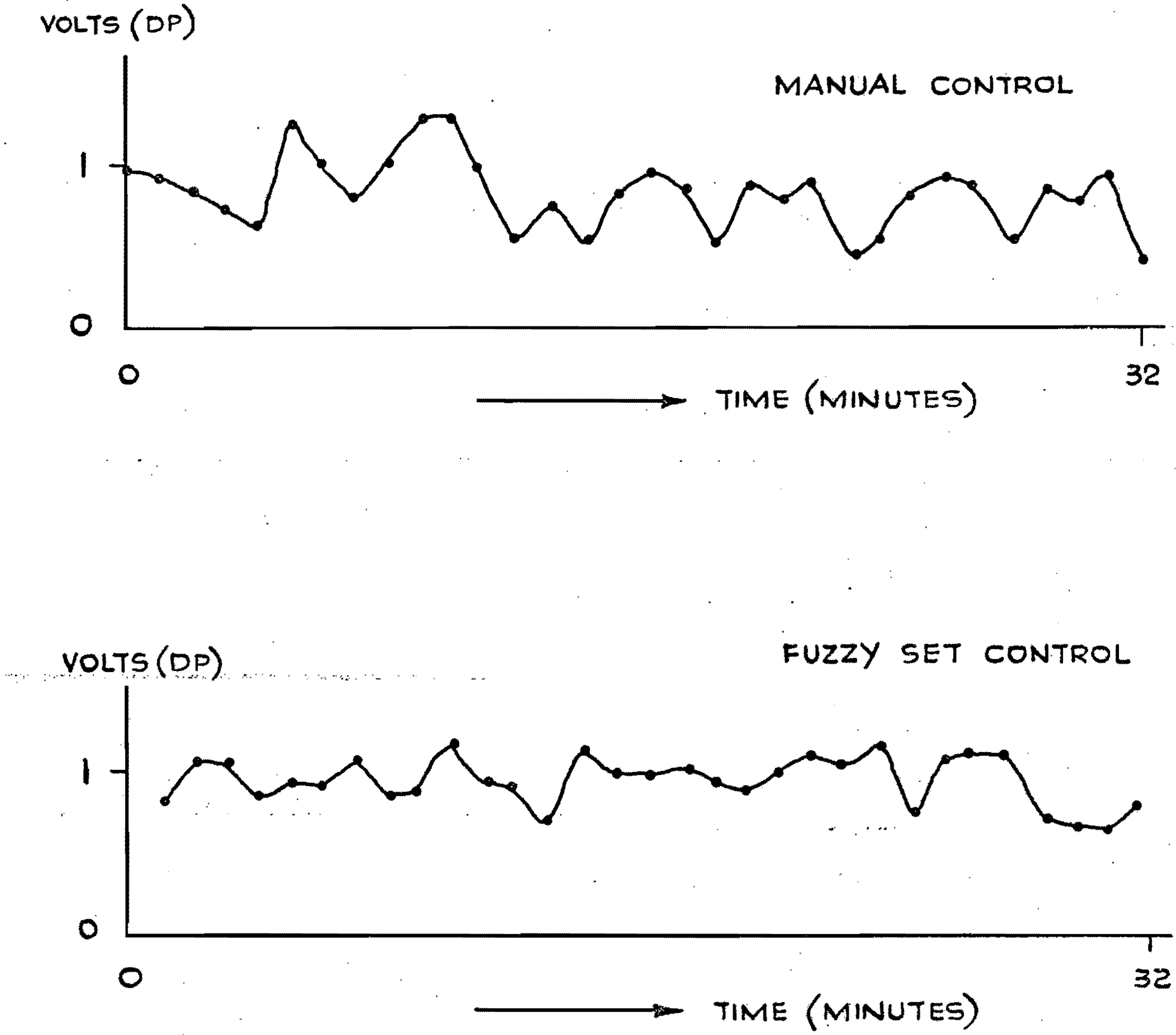
THE FOLLOWING CONTROL RULES ARE PRESENTED IN THEIR LINGUISTIC FORM TOGETHER WITH THE TABLE OF OUTPUTS THEY GENERATE

ALGORITHM 1

IF E IS PB AND S IS PB OR PM OR PS THEN W IS PB
IF E IS PB AND S IS ZE OR NS THEN W IS PM
IF E IS PB AND S IS NB OR NM THEN W IS ZE
IF E IS PM OR PS AND S IS PB OR PM THEN W IS PM
IF E IS PM AND S IS PS OR ZE THEN W IS PS
IF E IS PS AND S IS PM OR PS OR ZE THEN W IS PS
IF E IS ZE AND S IS PB OR PM THEN W IS PS
IF E IS ZE AND S IS PS OR ZE OR NS THEN W IS ZE
IF E IS ZE AND S IS NB OR NM THEN W IS NS
IF E IS NS AND S IS NM OR NS OR ZE THEN W IS NS
IF E IS NM AND S IS NS OR ZE THEN W IS NS
IF E IS NM OR NS AND S IS NB OR NM THEN W IS NM
IF E IS NB AND S IS PB OR PM THEN W IS ZE
IF E IS NB AND S IS PS OR ZE THEN W IS NM
IF E IS NB AND S IS NB OR NM OR NS THEN W IS NB

S \ E	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
-6	-3	-3	-2	-2	-2	-2	-1	-1	0	0	0	0	0
-5	-3	-3	-2	-1	-2	-2	-1	-1	0	0	0	0	0
-4	-3	-3	-2	-1	-1	-1	-1	-1	0	1	0	0	0
-3	-3	-2	-1	-1	-1	-1	0	0	0	1	1	1	1
-2	-3	-3	-1	-1	-1	-1	0	0	0	1	2	2	2
-1	-3	-3	-1	-1	-1	-1	0	0	1	1	1	2	2
0	-2	-2	-1	-1	-1	-1	0	1	1	1	1	2	2
1	-2	-2	-1	-1	-1	0	0	1	1	1	1	3	3
2	-2	-2	-2	-1	0	0	0	1	1	1	1	3	3
3	-1	-1	-1	-1	0	0	0	1	1	1	1	2	3
4	0	0	0	-1	0	1	1	1	1	1	2	3	3
5	0	0	0	0	0	1	1	2	2	1	2	3	3
6	0	0	0	0	0	1	1	2	2	2	2	3	3

FIG. 3.



TYPICAL CURVES OF PERMEAMETER DIFFERENTIAL PRESSURE VERSUS TIME BOTH IN MANUAL AND AUTO CONTROL

FIG. 4.

TRIAL NUMBER	TYPE OF CONTROL	SD OF DIFFERENTIAL PRESSURE (VOLTS)	% REDUCTION IN SD (V.~.MANUAL)	DATE OF TRIAL
1.	MANUAL	0·27	-	8 SEPT. '75
	FUZZY SET	0·16	40%	
2.	MANUAL	0·26	-	23 SEPT. '75
	FUZZY SET	0·20	23%	
	FUZZY SET (÷2)	0·16	38%	
	FUZZY SET (×2)	0·27	-4%	
3.	MANUAL	0·18	-	2 OCT. '75
	TWO TERM CONTROL	0·12	33%	

TABLE SHOWING RECORDED CHANGES IN STANDARD DEVIATION OF PERMEABILITY FOR DIFFERENT MODES OF CONTROL

TABLE 1.

CONTROL OF COMPLEX SYSTEMS BY FUZZY LEARNING AUTOMATA

by

Y.M.El-Fattah

1. AIMS OF THE PROJECT:

The aim of the project is to search for new methods for control of complex systems, where the goals , the constraints, and the consequences of possible actions are too ill defined or complex to admit of precise or conventional mathematical analysis. It is hoped that the research results could be applied to a specific technological process.

2. METHODS OF THE PROJECT:

The theory of fuzzy sets and automata will be mainly employed in that search. The research will parallel the development of learning control systems which has traditionaly drawn on Markov processes, statistical decision theory, automata theory, hill-climbing techniques, information theory, and pattern recognition,cf. e.g. Y.M. EL-FATTAH [1], [2], K.S.FU [3] , L.M.LYUBCHIK and A.S.POZNYAK [4] . The search will attempt to extend and elaborate on the results of related works like S.S.L.CHANG and ZADEH [5] W.G.WEE and K.S.FU [6] , K.ASAI and S.KITAJIMA [7] , L.A. ZADEH [8] , to mention just a few .

3. OUTLINE OF THE ANALYSIS:

Let $U \subset R^m$ be the set of feasible controls, and $Y \subset R^n$ be the set of system outputs. At any time step t denote the control input and the system output by $u(t)$ and $y(t)$, respectively. The next system output $y(t+1)$ is considered as a fuzzy set $p(t+1)$ on Y. The controlled system is represented by a fuzzy mapping [5] from $Y \times U$ into Y,

$$f: \quad Y \times U \quad \to \quad Y$$

$$\mu_{p(t+1)}(y(t+1)) = \mu_f(y(t),u(t);y(t+1)). \tag{1}$$

The control policy is represented by a fuzzy mapping from Y into U

$$g: \quad Y \quad \to \quad U$$

$$\mu_{q(t+1)}(u(t+1)) = \mu_g(y(t);u(t+1)) \tag{2}$$

The control $u(t+1)$ is represented as a fuzzy set $q\ t+1)$ on U.

We carry out the discretization of the domain of outputs Y and controls U into some sets of subdomains $\{Y_i\}$ and $\{U_i\}$ respectively such that

$$Y_i \neq \Theta\ ,\ Y_i \cap Y_j = \Theta\ (i \neq j),\ Y_i \subset Y,\ \bigcup_{i=1}^{s_1} Y_i = Y,\ (i,j=1,..,s_1)$$

$$U_i \neq \Theta\ ,\ U_i \cap U_j = \Theta\ (i \neq j),\ U_i \subset U,\ \bigcup_{i=1}^{s_2} U_i = U,\ (i,j=1,..,s_2) \tag{3}$$

where Θ is the null set.

As a model of the controlled system being investigated we propose to use a fuzzy learning automaton [6] , [7] , i.e. a fuzzy automaton for which the membership functions which are entries of the transition matrix are modified by a suitable learning operation.

A fuzzy learning automaton is also considered as a model of the controller. The automaton updates the entries of the fuzzy mapping "g" each time information is fedback from the controlled system and its model, see Fig.1.

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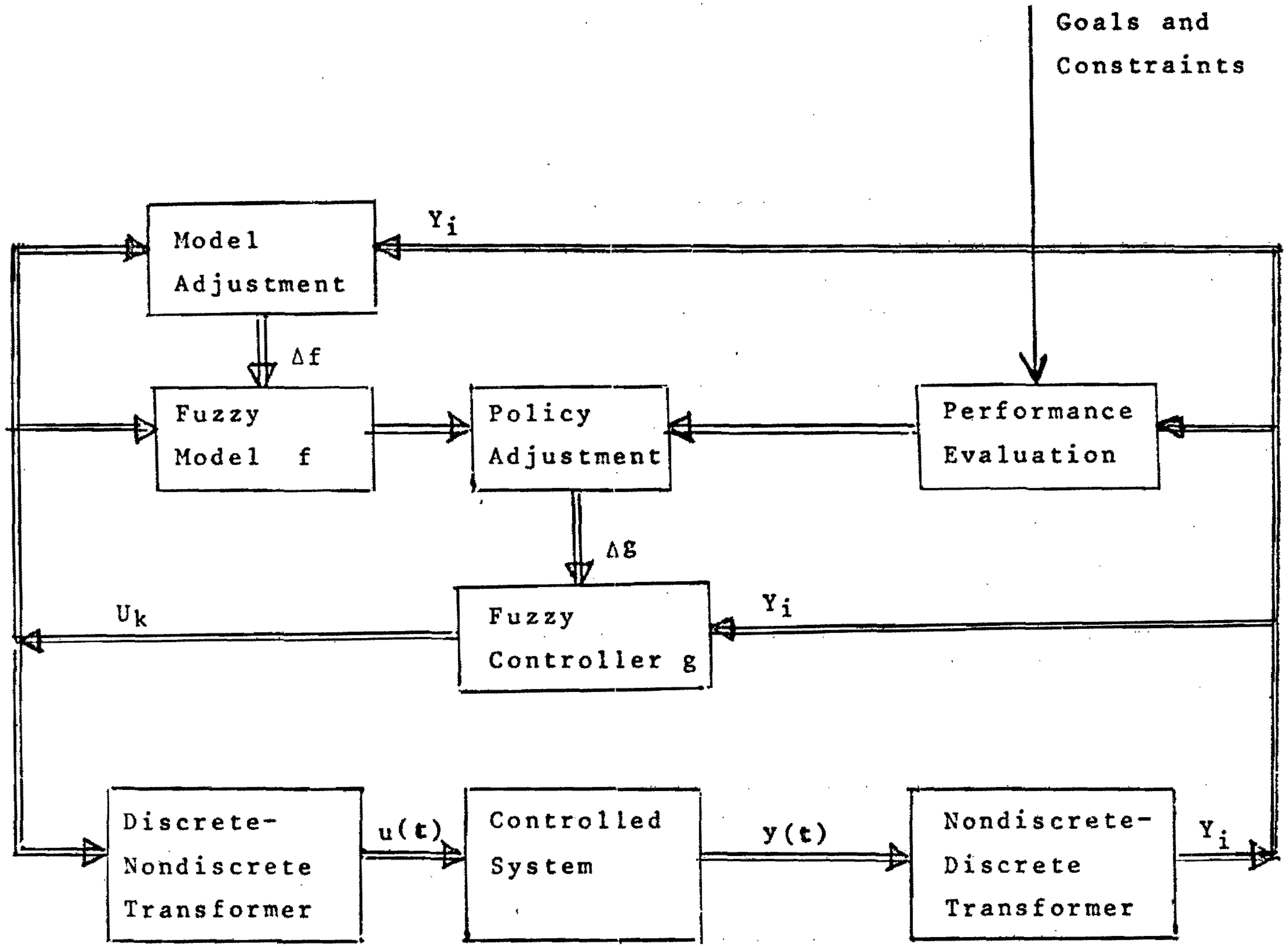

Goals and
Constraints
Model
Adjustment
Yi
Δf
Fuzzy
Model f
Policy
Adjustment
Performance
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Δg
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Controller g
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Discrete-
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y(t)
Nondiscrete-
Discrete
Transformer
Yi

Fig.1.

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MULTIVALUED LOGICS AND FUZZY REASONING

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1 Shallow and Exact Reasoning

These notes are concerned with recent developments in multivalued logic, particularly in fuzzy logic and its status as a model for human linguistic reasoning. This first section discusses the status of formal logic and the need for logics of approximate reasoning with vague data. The following sections present a basic account of fuzzy sets theory; fuzzy logics; Zadeh's model of linguistic hedges and fuzzy reasoning and finally a bibliography of all Zadeh's papers and other selected references.

Models of the human reasoning process are clearly very relevant to artificial intelligence (AI) studies. Broadly there are two types: psychological models of what people actually do; and formal models of what logicians and philosophers feel a rational individual would, or should, do. The main problem with the former is that it is extremely difficult to monitor thought processes - the behaviorist approach is perhaps reasonable with rats but a ridiculously inadequate source of data on man - the introspectionist approach is far more successful (e.g. in analysing human chess strategy) but the data obtained is still incomplete and may not reflect the actual thought processes involved.

Formal models of reasoning avoid these psychological problems and have the attractions of completeness and mathematical rigour, hopefully proving a normative model for human reasoning. However, despite tremendous technical advances in recent years that have greatly increased the scope of formal logic, particularly modal logic (Snyder 1971), the applications of formal logic to the imprecise situations of real life are very limited. Some 50 years ago, Bertrand Russell (1923) noted:

"All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence ..... logic takes us nearer to heaven than other studies".

The attempts of logicians to rectify this situation and broaden the scope of logic to cover various real-world problems has been surveyed recently by Haack (1974), and the role of modern developments in philosophical logic in AI has been excellently presented by McCarthy & Hayes (1969). These present notes are concerned with an area of massive recent development not covered by either of these references, that of 'fuzzy logic' and approximate reasoning initiated by Lofti Zadeh.

It is no coincidence that Zadeh's previous work had been concerned with successively improved refinement in the definitions of such terms as 'state' and 'adaptive' in systems engineering. It was dissatisfaction with the decreasing semantic content of such increasingly refined concepts that led to his (1972 "Fuzzy languages") remarks that:

"In general, complexity and precision bear an inverse relation to one another in the sense that, as the complexity of a problem increases, the possibility of analysing it in precise terms diminishes". "Thus, 'fuzzy thinking' may not be deplorable, after all, if it makes possible the solution of problems which are much too complex for precise analysis".

During recent years Zadeh (see bibliography) has developed in detail a model for approximate reasoning with vague data. Rather than regard human reasoning processes as themselves "approximating" to some more refined and exact logical process that could be carried out perfectly with mathematical precision, he has suggested that the essence and power of human reasoning is in its capability to grasp and use inexact concepts directly. He argues that attempts to model, or emulate, it by formal systems of increasing precision will lead to decreasing validity and relevance. Most human reasoning is essentially 'shallow' in nature and does not rely upon long chains of inference unsupported by intermediate data - it requires, rather than merely allows, redundancy of data and paths of reasoning - it accepts minor contradictions and contains their effects so that universal inferences may not be derived from their presence.

The insight that Zadeh's arguments give into the nature of human thought processes and, in particular, to their support of replication in the computer, are of major importance to a wide range of theoretical and applied disciplines - particularly to the role of formalism in the epistemology of science. The arguments have become associated with 'fuzzy sets theory' (Zadeh 1965) and this does indeed provide a mathematical foundation for the explication of approximate reasoning. However, it is important to note that Zadeh's analysis of human reasoning processes and his exposition of fuzzy sets theory are not one and the same - indeed they are quite distinct developments that must be separated, at least conceptually, if a full appreciation is to be had of either. As analogies one may conceive that fuzzy sets are to approximate reasoning what Lebesgue integration is to probability theory; what matrix algebra is to linear systems theory; or what lattice theory is to a propositional calculus.

The table below was compiled from an up-to-date bibliography on fuzzy systems containing some 300 references (Gaines & Kohout 1976) and demonstrates the growth of such work in recent years:

65	66	67	68	69	70	71	72	73	74	75
2	5	4	11	16	17	31	46	58	64	31 (at May 75)

Table of papers on fuzzy systems by year of publication

The relevance of this work to AI is indicated by its many recent applications to subject areas such as: pattern recognition (Siy & Chen 1974); taxonomic clustering (Bezdek 1974); analysis of line drawings (Chang 1971); robot planning (Goguen 1974, Kling 1973, LeFaivre 1974); medical diagnosis (Albin 1975); engineering design (Becker 1973); systems modelling (Fellinger 1974); process control (Mamdani & Assilian 1975); and management information systems (Wenstop 1975). The remainder of these notes are concerned with fuzzy sets theory, fuzzy reasoning, and its relations to developments in multivalued logics.

2 Fuzzy Sets Theory

Zadeh (1965) first developed the concept of a fuzzy set as an extension of that of a standard set in which the characteristic function, $A(x)$, of an element, x, of a set, A, was allowed to take not only the values 0 (not a member) and 1 (a member), but also to range anywhere between these values - the semantics were to be consistent with the natural order on the unit interval, e.g. that $A(x)=0.6$ denotes a greater 'degree of membership' than does $A(x)=0.4$. To correspond to the natural concepts of intersection and union it would be expected that the degrees of membership to fuzzy subsets, A and B, would not be decreased in their union nor increased in their intersection. Zadeh postulates that the resultant values are the lowest and the highest possible, respectively:

$$C = A \cup B \rightarrow C(x) = \max(A(x), B(x)) \qquad (1)$$

$$C = A \cap B \rightarrow C(x) = \min(A(x), B(x)) \qquad (2)$$

It remains to define the complement of a fuzzy set, and Zadeh postulates that:

$$B = \bar{A} \rightarrow B(x) = 1-A(x) \qquad (3)$$

All these definitions reduce to the standard case when the characteristic function is restricted to its usual binary values. However, it would be fallacious to assume that the extension outlined is the only one with this property. For example, whilst the definitions of union and intersection use naturally defined extreme values, that of negation may seem more arbitrary. Any antitone mapping of the unit interval into itself that inverted 0 & 1 would also be consistent with both the binary case and the semantics of the ordering of truth values. For example, an alternative negation, A, might be:

$$B = \hat{A} \rightarrow B(x) = \begin{cases} 1 \text{ if } A(x)=0 \\ 0 \text{ otherwise} \end{cases} \qquad (4)$$

This has the property that, in general, $\hat{\hat{A}} \neq A$, which is desirable in modelling the intuitionistic propositional calculus (IPC - section 3.1) where inferences from negative data are disallowed. Zadeh has discussed alternatives to definitions (1) through (3), as have many other authors - the particular 'max' and 'min' rules of fuzzy sets theory are not fundamental to its application to approximate reasoning. However, they are the most widely used bases for fuzzy logic in the literature.

Given these basic definitions it is possible to 'fuzzify' any domain of mathematical reasoning based on set theory by assuming that variables do not take specific values but instead have a separate 'degree of membership' to each possible value. That is, instead of having a sharp value, a variable is fuzzily restricted to a domain of values. The definition of the 'value' of a function of many variables may now be extended to fuzzified variables in a natural way - if in the standard case y=f(x1,x2,...), and u(x1) is the degree of membership of a particular 'value' to x1, then:

$$u(y) = \begin{cases} \underset{x}{\text{MAX}}\ (\min(u(x1),u(x2),..)) \\ 0, \text{ if no x exists} \end{cases} \quad (5)$$

where x=(x1,x2,...) is any n-tuple such that y=f(x1,x2,...). That is: with each argument to the function is assoicated a degree of membership that is the lowest of those of each of its components; and with each value of the function is associated a degree of membership that is the highest of all the arguments resulting in that value.

In the same way that probability distributions are normalized to sum to unity and this is preserved under transformations, there is a natural normalization of the degrees of membership of a variable that is preserved under fuzzification. A fuzzy variable is said to be 'normalized' if at least one value has a degree of membership of unity. It is readily seen that a function, fuzzified as in equn.(5), of normalized variables is itself normalized (there must be at least one argument with degree of membership 1 and this will give a value with the same membership).

Zadeh's 1965 paper was presented as an extension of set theory and there has been a great deal of literature concerned with the technicalities of fuzzifying various mathematical structures, topologies, automata, etc., and determining what theorems remain proveable in the essentially generalized structure. Such work underpins the foundations of any future applications of fuzzy sets theory and is included in the bibliography. However it is the semantics of the theory applied to vague reasoning that there is most of relevance to AI.

3 Fuzzy and Other Multivalued Logics

Any logical structure may be fuzzified by considering propositions to have degrees of membership to truth values. If we take the conventional propositional calculus (PC) with truth values 0 & 1, then after fuzzification each statement, A, will be represented by a pair of values, (a1,a2), representing its degree of membership to falsity and truth, respectively. For example, fuzzifying the truth table for implication, $\supset$, in PC gives the following expression:

If C = A$\supset$B then (c1,c2)=(min(a2,b1),max(min(a1,b1),min(a1,b2),min(a2,b2))) (6)

Similar expressions may be derived for fuzzifying the truth tables of negation, $\sim$, disjunction, $\vee$, conjunction, $\wedge$, and equivalence, $\equiv$, but they are more meaningfully obtained by noting that fuzzification preserves the relations giving interdefinability of the connectives of PC. That is, if F is any false proposition (i.e. (f1,f2)=(1,0)), then we may write:

$$\sim A \quad \text{for} \quad A \supset F \tag{7}$$

$$A \vee B \quad \text{for} \quad \sim A \supset B \tag{8}$$

$$A \wedge B \quad \text{for} \quad \sim(\sim A \vee \sim B) \tag{9}$$

$$A \equiv B \quad \text{for} \quad (A \supset B) \wedge (B \supset A) \tag{10}$$

Equn.(7), for example, when substituted in (6) gives us:

$$\text{If } B = \sim A \text{ then } (b1,b2)=(a2,a1) \tag{11}$$

and, similarly, expressions may be derived for the other connectives.

If we assume the fuzzy variables are normalized then, as there is only one non-zero component, there is a 1-1 correspondence with the unit interval that simplifies the above expressions. Let:

$$a = (1-a1+a2)/2 \tag{12}$$

and so on for the other variables (this transformation can be inverted given that one of a1 and a2 must be 1). Then the equations for the logic operations become:

$$C = A \supset B \quad \rightarrow \quad c = \max(1-a,\ b) \tag{13}$$

$$B = \sim A \quad \rightarrow \quad b = 1-a \tag{14}$$

$$C = A \vee B \quad \rightarrow \quad c = \max(a,b) \tag{15}$$

$$C = A \wedge B \quad \rightarrow \quad c = \min(a,b) \tag{16}$$

$$C = A \equiv B \quad \rightarrow \quad c = \min(\max(1-a,b),\max(1-b,a)) \tag{17}$$

This set of simpler equations is what a number of authors have proposed as a 'fuzzy logic' (e.g. Lee 1972), probably not deriving them as a fuzzification of PC but instead as a direct set-theoretic interpretation of a logic based on equns.(1) through (3). The relation between equns.(3) & (14) is particularly interesting since fuzzification does not involve the complement operation, and hence the coincidence of definitions shows that Zadeh's definition of a fuzzy complement is a natural one for PC.

3.1 Relationship to VSS and Godel & Lukasiewicz Logics

Equns.(15) & (16) are valid for the disjunction and conjunction connectives of a wide range of multivalued logics (Rescher 1969), and it is interesting to examine the relationship of the system of equns.(13) through (17) to such logics. It turns out to be identical to the infinitely valued version of the 'variant-standard sequence' (VSS) investigated by Dienes (Rescher 1969 p.49) - i.e. VSS is exactly the fuzzification of PC. This logic has a defect in its semantics of inference, as noted by Lee (1972), that the assertion that A implies B (with value 1) does not necessitate that $b \geq a$, the truth value of B is greater than or equal to that of A. This seems a natural requirement in terms of our interpretation of the natural ordering of 'degrees of membership', and is implicitly assumed in most practical applications of fuzzy logic (e.g. Mamdani & Assilian 1975). It enables the assertion of a rule of the form, $A \supset B$, to be interpreted that B has a truth value in a particular instance at least equal to that of A, and hence greater than or equal to the maximum of any A1, A2, etc., that imply B.

If we require that the truth value of $A \supset B$ is 1 when $b \geq a$ then this may be used to define a variant of VSS based on some subset of definitions (9) through (17). To complete the definition of implication we must define the truth value of $A \supset B$ when $b < a$. Two possible definitions are:

$$C = A \supset B \rightarrow c = 1 \text{ if } b \geq a, \quad c = b \text{ otherwise} \tag{18}$$

$$C = A \supset B \rightarrow c = 1 \text{ if } b \geq a, \quad c = 1-a+b \text{ otherwise} \tag{19}$$

so that, when the implication is not absolute, the truth value is that of the implied proposition (equn. 18), or (equn. 19) it is a function of the difference between the two. If we couple each of these definitions with (7) for negation, (10) for equivalence, (15) for disjunction, and (16) for conjunction, we get two important systems: equn.(18) gives Godel's infinitely valued logic (Rescher 1969 p.45) which has a negation similar in form to the complement of equn.(4) and is closely related to the intuitionistic propositional calculus; equn.(19) gives Lukasiewicz's infinitely valued logic (Rescher 1969 p.37) which is the one used by Zadeh for statements involving truth and falsity in linguistic reasoning.

3.2 Relationship to Probability Logic

Other multivalued logics, some with connectives other than those of equns.(15) & (16) for disjunction and conjunction, may be derived from other subsets of these definitions - only the semantics of particular classes of situation can determine whether one system is more appropriate than another. The only other one to which I shall draw attention is that of 'probability logic' (PL). Rescher (1969) shows that the standard axioms for unconditional probability may be regarded as defining a logic which is closely related to the modal logic S5. PL is not truth-functional in that the truth value of a proposition is not uniquely defined by those of its components. Gaines (1975 "Stochastic ...") has shown that PL may be made truth functional in two distinct ways: (a) By assuming statistical independence between atomic components, a common assumption in systems engineering; (b) By assuming that of any two atomic components one must imply the other, giving a fuzzy logic satisfying equns.(15) & (16).

The equivalents of equns.(15) & (16) for a PL with assumed statistical independence are:

$$C = A \vee B \rightarrow c = a+b-ab \tag{20}$$

$$C = A \wedge B \rightarrow c = ab \tag{21}$$

Gaines (1975 "Stochastic ...") has re-analysed Mamdani & Assilian's (1975) data on experiments with a fuzzy logic linguistic controller using this form of connective and shown that it makes no difference to the results - the 'fuzzy reasoning' used is robust to changes in the form of 'fuzzy logic' on which it is based (more information on this controller is given in my notes on "Control Engineering & AI").

Giles (1975) has given a model for various forms of multivalued and probability logics as a dialogue between two participants, in essence a game-theoretic semantics. Gaines (1975 "Fuzzy ...") has given an alternative model that also encompasses both probability and fuzzy logics in terms of the responses of a population (e.g. people or neurons). Atomic propositions are modelled as questions to which each member of the population makes a binary, yes/no, response - the truth value of a proposition is the proportion of 'yes' responses, and that of compound propositions is determined by counting those who say yes to both A & B for terms of the form, $A \wedge B$, and so on for more complex compounds. This is essentially a set-theoretic model of a general logic and different specialized forms may be obtained by adding further constraints to it:

(i) If we assume that a 'yes' to A implies a 'no' to ~A then we obtain Rescher's probability logic;

(ii) If further we assume that the responses are independently distributed in the population we obtain what Gaines (1975) terms a 'stochastic logic' satisfying equns.(20) & (21);

(iii) If we assume instead that members of the population each evaluate any questions according to the same criteria but each require a different, individual 'weight of evidence' to reply 'yes', then we obtain a fuzzy logic satisfying equns.(15) & (16).

This last assumption, so different from the conventional one of statistical independence, also has its intuitive attractions. Reason (1969) has shown that the threshold applied by people in coming to a binary decision on an essentially analog psychophysical variable seems to be associated with personality factors and is characteristic of the individual. If so, human populations would tend to show a more fuzzy than stochastic logic in their overall decision making. Similarly the concept of uniformity in information-processing but varying thresholds of sensitivity is a reasonable one for populations of cells. Note that both the Giles and Gaines models give the pure forms of the logics as extreme cases - the most reasonable general assumption is a mixed form of probability/fuzzy logic.

Thus developments in 'fuzzy logic' and 'fuzzy reasoning' may be related both to classical multivalued logics and to classical probability theory. One suspects that there must be some underlying unifying structure that would form a better basis for modelling human reasoning than any of these particular logics alone - certainly no one of them has a claim at present to be the one correct logic for reasoning under uncertainty.

4 Linguistic Variables, Hedges and Fuzzy Reasoning

Whilst the technical aspects of both fuzzy sets and fuzzy logics have attracted much attention and are fascinating and significant in their own right, it is in their application to linguistics and approximate reasoning that their practical importance lies. It is not possible to do justice in these notes to Zadeh's prodigious output and detailed arguments, or to the application studies of recent years. The following extracts are intended to give a feel for the approach and motivate further reading of the literature in the bibliography. A good general introduction is given in Zadeh (1973 "Outline of ..."); Lakoff (1973) gives a linguistic introduction; Goguen (1974) is more technical but relates categories and concepts; Kling (1973) and Lefaivre (1974) have developed a version of planner capable of fuzzy reasoning; Albin (1975) and Wenstop (1975) have used models of fuzzy reasoning in studies of medical diagnosis and management information systems, respectively; and so on - the subject area now has a high semantic content in addition to its technical attractions.

Three illustrations will serve to define the type of problem with which Zadeh is concerned:

(1) Reasoning with 'linguistic variables' such as: "young", "middle-aged", "tall", or "rich", rather than precise quantities such as: "12 years old", "45 years old", "6 feet tall", or "having $1M";

(2) The effect of general linguistic 'hedges' upon such variables, e.g. "very small", "more or less tall", "fairly rich", etc., which allow a single concept to be extended in a standard way to cover many more situations;

(3) Syllogisms for approximate reasoning with linguistic variables, e.g. "John is very old - Charlie is about the same age as John - so Charlie is old".

Zadeh represents the meaning of a linguistic variable as a 'compatibility function' or 'fuzzy restriction' assigning a degree of membership to each possible value of the variable. For example, "older" might correspond to degrees of membership commencing at 0 for age 0 and increasing very slowly to 0.1 at age 25, to 0.3 at age 40, and then more rapidly to 0.9 at age 65, and then more slowly, asymptotic to unity. The numerical forms of such functions do not matter a great deal since it is the order relations that play most part in the later development. MacVicar-Whelan (1974) has performed some psychological experiments on their form and Lakoff (1973) reports similar experiments. Individuals do find it natural to assign such numerical values to the degree of compatibility of a particular value with a concept. Alternatively one may think of a population model in which the compatibility is measured in terms of the proportion of people who say, "yes, a young man may be 25 years old". Many models are possible and it is useful to have one in mind, but again much of the development of a theory of linguistic reasoning is independent of the exact model.

Zadeh has given a detailed account of how, given the compatibility function for a single linguistic variable such as "young", the compatibility functions may be calculated for the same variable subject to linguistic hedges, "not very young", "more or less young", etc. He shows how complex hedges may be decoded by a standard syntax into a number of elementary operations on compatibility functions and gives approximate forms of such operations as arithmetic operators. These definitions give a superficial appearance of mathematical precision to the effect of hedges. However Zadeh introduces the notion of 'linguistic approximations' in which compatibility functions resulting from a process of fuzzy reasoning are described by the closest reasonably simply-hedged linguistic variable. This process means that the reasoning itself is essentially approximate, 'shallow reasoning' that loses information at each stage, and may therefore consist only of comparatively short chains.

As noted in Section 3, the logic which Zadeh chooses to fuzzify for linguistic statements involving truth or falsity is one of Lukasiewicz's multivalued logics with connectives defined by equns.(10), (14), (15), (16) & (19). Hence the form of implication used is not that of PC which, when fuzzified, gives counter-intuitive results. This is not really surprising in that there are philosophical objections to the implication of PC as an explication of "if ... then" in ordinary language. Lukasiewicz originally developed his logic in 3-valued form to allow for the status of future contingent propositions, and later extended it to have the semantics of a "modal" logic.

5 Conclusions and Background References

The classical formal logics such as PC may be seen as expressing idealized, precise 'reasoning', such as that of the digital computer at a hardware logic level. AI research may be seen as an attempt to replicate the less formal linguistic reasoning with vague and imprecise rules and data, actually adopted by human beings. This is not in itself a new problem - in "A System of Logic" published in 1843, John Stuart Mill commences with the remark:

"Since reasoning, or inference, the principal subject of logic, is an operation which usually takes place by means of words, and in complicated cases can take place in no other way: those who have not a thorough insight into both the signification and purpose of words, will be under chances, amounting almost to certainty, of reasoning or inferring incorrectly".

(The rest of this fascinating book is also worth reading - there are few problems of knowledge and its acquisition about which Mill has no perceptive comments - it is a pity that he did not have access to a PDP10 !). He criticizes the weakness of formal logic in explicating linguistic reasoning but, like most work since, attempts to bridge the gap linguistically rather than develop a new basis in logic. Zadeh's use of fuzzy logic to model natural linguistic reasoning may be viewed as a more direct response to Mill's argument above some 130 years later.

Apart from papers so far reference, I would recommend anyone interested in this area to have at hand: Rescher's (1969) book on multi-valued logics; Snyder's (1971) book on modal logics as an introduction and Hughes & Creswell (1968) as a reference; Creswell's (1973) book on logic and language as an alternative modern approach to linguistic semantics; Fillmore & Langendoen (1971) and Hockney et al (1975) as basic references on the same; and Krantz et al (1971) for alternative approaches to partially qualitative description. McCarthy & Hayes (1969) is well worth reading first, followed by Lakoff (1973) and any (or all !) the Zadeh references.

Having quoted so many eminent authorities I may as well end with a quote from the most venerable of them all - Lazarus Long, the senior, was over 1,000 years old when he wrote:

"The difference between science and the fuzzy subjects is that science requires reasoning, whilst those other subjects merely require scholarship" (R. Heinlein, "Time Enough for Love", NEL 1974).

Hopefully the direction of the work described in these notes indicates that the scholarship of multivalued logic has a part to play in the science of reasoning about (rather fuzzy) human linguistic behaviour!

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WHY FUZZY REASONING ?

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1 Background

The topics of "fuzzy logic" and "fuzzy reasoning" are not clear-cut subject areas with well-defined results and track records. Instead they represent a wealth of recent activity on an international front that may be seen to have its technical roots in philosophical and mathematical studies of "multi-valued logics" (Rescher 1969) and "vague reasoning" (Machina 1974), but which owes much of its present impetus to engineering interest from those concerned with "information systems" (see Sanford 1975 for some wry comments on this "engineering interest" in a philosophical journal).

Much of the current literature on fuzzy logic is neither precise in its objectives nor accurate in its conclusions. Much of the current effort duplicates activities taking place, or having taken place, elsewhere. However, this is of the nature of a fast growing subject area - it makes it difficult, however, for the newcomer to assimilate the (literally hundreds) of papers of recent years and assess the results, neither dismissing them because of his contact with the trivial, nor believing the exaggerated claims of enthusiasts.

This seminar is intended to introduce this area, relate it to other subject areas concerned with reasoning and decision-making, and give pointers to the most useful literature and areas of development.

These notes are complementary to those on "Multi-Valued Logic and Fuzzy Reasoning" for the AISB Summer School (Gaines 1975), which gives a technical summary and literature references. I will only emphasize again that it is worthwhile commencing with Zadeh's papers and the more "philosophical" and "linguistic" literature that emphasizes the motivation behind the study of fuzzy reasoning rather than the more technical aspects of "fuzzy logic".

2 Basic Problems of Knowledge and Prediction

Because of the fuzzy nature of the subject area I feel one should go back to some fundamental considerations. These have massive and ancient philosophical roots. However, they are also of direct practical relevance - whenever we attempt to implement, for example, a management information system that does more than store and reproduce the data fed to it, to make inferences or estimate trends, we are involved in basic problems of knowledge whose "solution" entails assumptions - if we become concerned with the nature and reasonableness of these assumptions then we very rapidly come to face problems that have been the subject of philosophical debate for all recorded time.

However, our own attitudes to these problems have probably been formed in the light of the past century of the growth of science and the success of technology based on it. This places great emphasis on precise physical laws framed in terms of relations between numeric quantities. It has little use for human opinion and belief, and its development through verbal qualitative reasoning. Thus, when faced with problems of aiding the manager in decision-making we automatically fall back on probability theory based on measure theory and the observation of frequencies. This is not necessarily a natural tool in which to formulate the decision processes used by human beings. Work on fuzzy reasoning is best seen as stimulated by the quest for more natural tools in which to develop information systems that interface naturally with the human reasoning process.

2.1 Induction and Prediction

The purpose of reasoning is to draw inferences from established premises. It used to be thought meaningful to make a clear distinction between deductive reasoning in which the conclusions were logically derivable from the premises (and hence had no more content than them, were in essence a re-formulation), and inductive reasoning in which the conclusions involved an alogical inductive "leap" or generalization - the former was mathematically rigorous and the latter metaphysically dubious. This distinction attained its strongest form with Hume's (Popper 1972) (irrefutable) proof that the process of inductive reasoning cannot itself be proven valid.

This result may be seen as undermining any possible foundations of "science", and has naturally generated an immense effort among philosophers of science (such as Carnap (Carnap and Jeffrey 1971), Popper (1972), Lakatos (Lakatos and Musgrave 1970), Feyerabend (1975), Hesse (1974) and Gellner (1974)) to determine what are the foundations of science and to give them whatever lesser rigour Hume's result still permits. The relevant literature on the problem of induction (Katz 1962), confirmation theory (Swinburne 1973, Rescher 1973) and scientific inference (Hesse 1974) is important to anyone developping information systems. However, they will be disappointed at the strength of the negative results and the paucity of positive methodology.

More recently doubt has been thrown on the strength of deductive inference (Dummett 1973). Firstly, the whole concept of an established premise is extremely dubious. Even "raw observation" seems always to entail inductive reasoning - we cannot perceive or measure without unverifiable assumptions. Secondly, the uniqueness and absoluteness of classical logics (propositional and predicate calculi) has been increasingly challenged with increasing success (Haack 1974). In recent years the rigorous development of modal logics (Snyder 1971), the weakness of the classical logical foundations of quantum physics (Mehra 1973), the success of alternative logical calculi as foundations of mathematics (Mostowski 1966), and, probably also, the obvious poverty and weakness of our whole knowledge of knowledge, its acquisition and use, as demonstrated by the attempts to use it operationally in artificial intelligence systems - all have weakened the position of classical deductive reasoning.

3 Human Reasoning

Once we realize that any form of predictive inference involves alogical and unverifiable assumptions, that all premises have inherent vagueness if not some element of falsity, and that our process of reasoning, having papered over these basic flaws, is itself somewhat arbitrary, we must begin to wonder how anything is possible (or decide that in fact anything *is* possible - a perfectly tenable position if somewhat devastating for systems engineering!).

One natural way out is a form of pragmatism - "valid reasoning is what works". This is the argument that Hume proved circular - however, as Katz (1962) has argued, there is a difference between (logical) validation and (pragmatic) vindication. We may, for example, give various evolutionary arguments as to why creatures with brains whose reasoning is like our own have survived in this physical environment (what we cannot justify is the supposition that they will continue to do so - however, it seems reasonable to act as if this were so - at the utmost level of despair permitted to a working engineer one may operate under the motto of William the Silent, "It is not necessary to hope in order to act, or to succeed in order to persevere" !).

In the light of these strong undercurrents mining away the philosophical and methodological foundations of science, it is not surprising that one of the main pragmatic models of successful reasoning that is being examined is man himself. The last hundred years of scientific and commercial success of physical and mechanist science gave great hopes that such science would lead to a complete account of biological processes, including all aspects of the human brain and its reasoning capabilities. One would not look to the human mind as a model of inference processes - the precision and exactness of formal logical deduction are foreign to the forgetful, inexact, wandering human mind. Perhaps, conversely, creative and original thought was foreign to the precision of the digital computer, but the judicious introduction of "noise" might achieve it without necessarily introducing the basic weaknesses of the brain.

We would not nowadays wish to return to a position where the brain was regarded as having a vitalist component beyond our knowledge, nor the computer regarded as pre-programmed in every respect and thus incapable of the emulation of "creativity". We are making too much progress in understanding, emulating and collaborating with human reasoning to feel the need to invoke magic, and no-one who has retrieved interactively from a natural language data base system which has also interacted with other users (and contains data resulting from those interactions) could deny the creativity of some computer systems (constructive novelty is essentially always relative to the percipient - we are the ones who recognize innovation and what it is reflects upon both observer and observed).

However, there is an increasingly healthy respect for human reasoning that begins to recognize the problems of inferencing from unreliable, inconsistent and vague premises to conclusions that form the basis for action.

Perhaps forgetting, inexactness, and search for analogies, are not defects of a weak deductive system but instead essential features of a powerful inductive system (those who see this as the obvious position anyway should introspect a little more deeply and ask even if they believe it superficially do they actually act on it in systems engineering - we have been indoctrinated to believe in the superiority of numbers and exact operations to names and qualitative operations - this affects many design decisions - for example, we generally require far more precision of expression by the computer user than is necessary - we are surprised that alphabetic names can just as well be entered on a 10-key telephone dialler (with 2.6 to 1 vagueness) as on a teleprinter - we trust numerical approximations to reality and our manipulations of them far more than any direct verbal logic).

An interest in human (verbal) reasoning processes is not new - Plato and Aristotle had a lot to say that is still very fresh today. The modal logicians studying our use of terms such as "possible" and "necessary" (Snyder 1971), "sometimes" and "always" (Prior 1967), "a few" and "many" (Altham 1971), and so on have essentially modelled the reasoning processes of which these terms are major components. Both modal logic and linguistics have made great progress in this direction in recent years (Creswell 1973, Fillmore and Langendoen 1971, Hockney, Harper and Freed 1975). The technical development of fuzzy logic and fuzzy reasoning may be seen as providing enhanced mathematical tools for the study and emulation of human verbal reasoning, logics which carry both factual information are estimates of its reliability. Probably more important than any single technical development however is the motivation behind the surge of engineering interest in such logics - it has brought together many workers on diverse forms of information systems in the common realization that there are substantial gaps in our knowledge of knowledge that are being filled ad hoc in many practical systems and which need, and can sustain, far greater coherent development.

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RESEARCH NOTES ON FUZZY REASONING

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(1) Fuzzy Logic and Fuzzy Reasoning

There appears to be an important gap between the so-called "fuzzy logics" studied by many authors (e.g. Lee 1972 - JACM, Bellman and Giertz 1973, Inf. Sc.) and the "fuzzy reasoning" developed by Zadeh (Berkeley reports) which is not noted in the literature. The standard "fuzzy logic" is just a multi-valued logic of the family described by Rescher (1969 book). Most authors probably regard it as conjunction/disjunction/negation derived from fuzzy set membership considerations completely analogous to the derivation of Boolean algebra from classical set theory. Five points seem to have been missed:

(1.1) The definition of implication is open - it is not determined by the other connectives. For example, the natural definition of negation in terms of implication:

Def. $\bar{A}$: $A \supset F$

gives "fuzzy negation" ($u(\bar{A}) = 1-u(A)$) for both VSS implication ($u(A \supset B) = \max(1-u(A), u(B))$) and Lukasiewicz implication ($u(A \supset B) = \min(1, 1-u(A)+u(B))$).

Question 1 How many of Rescher's (see also Rosser and Turquette et al) logics have (or, if non-truth-functional, can be restricted to have) the fuzzy logic basic connectives ?

Question 2 What is the status of fuzzy negation, e.g. c.f. Gödel negation ?

(1.2) Some authors do not even consider implication - Lee assumes the PC definition:

Def. $A \supset B$: $\bar{A} \vee B$

without making this explicit. This then leads to the semantic inconsistency he notes for the relative magnitudes of the terms, but he does not follow this up to conclude that his definition is wrong, presumably because he does not realize he has one.

Question 3 What are the minimal semantic constraints upon the implication connective (see Carnap et al on confirmation theory) ?

(1.3) The "fuzzy logic" generally considered may be derived as the fuzzification of PC. This does not seem to have been stated explicitly but is probably folk lore. Such a derivation is more in the spirit of the general application of fuzzification to other mathematical structures.

(1.4) Fuzzified PC is precisely the variant standard system (VSS) described by Rescher (attributed to ?).

Question 4 Can other MV logics be derived as fuzzifications - e.g. Lukasiewicz and Godel (I doubt it: if not there may be some relationship to Dugundji's results on finite matrices) ?

(1.5) Zadeh himself does not seem to have put forward fuzzified PC (or rejected it because of the weakness of its implication connective ?). Fuzzy reasoning is primarily concerned with statements about fuzzy attributes - one possible attribute is a truth value. When Zadeh considers the truth value attribute he fuzzifies Lukasiewicz MV logic not PC.

Question 5 Why choose Lukasiewicz logic (Zadeh does not seem to give a justification) ?

(2) Fuzzified Lukasiewicz Logic (LL)

The similarity between the fuzzy set operations on degrees of membership and the LL basic connectives needs investigation (Quest. 4 and 5). There are semantic constraints upon the form of function that maps degree of membership onto truth value. These constraints should be made explicit. The logic without the constraints offers an apparent freedom that should be removed - this will itself lead to a different formulation.

Question 6 Given a family of functions on the unit interval (truth value → membership) and the operations of fuzzy sets theory and LL (or various subsets) what is the space of functions generated (and the converse of deriving a basis for a given space given the operations) ?

Question 7 What are the semantic constraints upon a basis ?

(3) Interaction of Truth Values with Fuzzy Statements

We may apply (essentially metalinguistic ?) statements about fuzzy truth values to statements about fuzzy attributes and there is the possibility of interplay between them. For example, what is the relationship between 'John is tall is very true' and 'John is very tall is true', or 'Mary is fat is more or less true' and 'Mary is more or less fat'. There is scope for interplay but no obvious rule. Notes:

(3.1) One must beware of unnatural examples and watch for the possibility of multiple interpretations - e.g. it does not seem to be meaningful to consider statements such as 'John is tall is .7 true', and 'John is tall is very true' could mean that he is so tall no one could disagree (in which case 'very tall' assumes a higher truth value) or that he is precisely tall, neither 'more or less tall', nor 'very tall' (in which case 'very tall' will assume a lower truth value).

(3.2) Ambiguities can only be resolved by accepting that the role of language is communication and that the same statement may have entirely different meanings for different recipients. The use of this feature of language is itself a major linguistic skill (essential to politicians).

(3.3) Our usage of words such as 'truth' and 'false' may be more related to 'reasonableness' and 'unreasonableness' than logical truth, i.e. a statement is very true because it is a reasonable way of expressing something that will create a very true impression of the state of affairs in the mind of the recipient. Thus 'John is tall is very true' would mean that it is the most reasonable statement to make about John's height. If he turns out to be 7 foot tall, you say 'but he is extremely tall, very very tall' and feel that you have been misled, i.e. 'John is tall' is 'not very true', but 'John is extremely tall' is very true.

This model of our use of the linguistic terms true and false in the metalinguistic context (i.e. about other statements) as relating to the communication of a true impression is a useful one, probably widely valid. It resolves the conflict between a direct interpretation of degrees of membership as degrees of truth - where 'John is very tall' makes 'John is tall' very true - and what seems to be the more conventional use of statements about truth and falsity in colloquial language. It also emphasizes that the analysis of linguistic interactions must be in a context of interpersonal communication, not isolated fragments.

(3.4) The use of linguistic hedges is not only to modify meaning but also to convey the level of precision. 'John is tall', 'John is more or less tall', 'John is pretty well tall', 'I think it is very true to say that John is tall', all convey the same expectation of height but varying degrees of possible spread about it. This is why a single truth value cannot express the full semantics of a vague statement.

(4) Linguistic Approximation Stable Fuzzy Arguments

Zadeh's concept of linguistic approximation (LA) introduces an element of discontinuity into the fuzzy reasoning process. LA arises basically because the numerical manipulations of fuzzy predicates corresponding to linguistic hedges and logical operations can generate a result that cannot be represented exactly as a simply hedged linguistic truth-value - it can only be approximated by one. The effect is similar to that of quantization in analog-digital conversion and generates similar problems, i.e. it cannot be treated effectively as "noise", introduces its own coloration, and gives rise to new phenomena such as limit cycles.

The importance of linguistic approximation to a theory of fuzzy reasoning seems to have been missed despite the emphasis Zadeh places upon it. Without it any form of fuzzy logic is a variant of some formal multi-valued logic and (whilst again the fact that Zadeh uses a fuzzified Lukasiewicz logic rather than fuzzified PC seems to have been little-noted) it is presumably open to axiomatization and probably to reduction to some known structure.

With LA fuzzy logic has new properties, for example that a long chain of reasoning that is logically equivalent to a shorter chain will produce less sharp results in general.

Several questions are apparent:

Question 8 What class of operation on fuzzy variables leads to a finite set of values ? - a purely technical point reducing the need for LA.

Question 9 Does LA account for the weakness in long chains of reasoning ?

We may introduce the concept of a stable fuzzy argument which is such that if LA is applied at all or any points in the chain of reasoning the LA to the final result is unchanged. This introduces the concept of a linguistic confluence set - the set of all possible results of a chain of fuzzy reasoning when LA is applied in all possible ways.

The following results are obvious: (a) the longer the chain of argument the less stable it will be; (b) the greater the range of LA's available the more stable it will be - this corresponds to the eskimos 40 names for ice, the skilled practitioner's use of longer chains of argument, etc.

LA introduces tolerance relations on the space of functions over an interval. Can we take the logic and tolerance relation and treat it as a new logic ?

(5) Fuzzified Definitions of System Concepts

It should be possible to re-develop such concepts as stability, adaptivity and state within a framework of fuzzy reasoning. Some of the arbitrariness in current definitions should be absorbed into the fuzziness rather than left as firm but undefined decisions. The semantic constraints that mean that decisions are not completely arbitrary will appear as the order relations on fuzzy values.

(6) The Role of the Numbers

How much of the theory of fuzzy reasoning can be developed in terms of order relations on degrees of membership rather than truth values. I doubt that this has been studied in the light of Zadeh's semantics for fuzzy reasoning, e.g. with fuzzified LL.

Linguistic Approximation in an order structure would give a tolerance leading to a non-truth-functional logic. This seems a very natural structure that is worth developing.

These notes were based on discussions with Lotfi Zadeh at Berkeley in May 1975.

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APPLICATION OF A FUZZY CONTROLLER IN A WARM WATER PLANT

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Summary

In many cases a human operator is far more succesful at controlling a complex industrial process than any controller derived using modern control techniques. The method of expressing the strategy of a human operator using fuzzy set theory has been proposed elsewhere. In this study this method is applied to the control of a warm water plant. Fuzzy algorithms based on linguistic rules describing the operator's control strategy are constructed to control this plant. Several types of such algorithms are implemented and compared with each other, in behaviour as well as in structure.

1 Introduction

Fuzzy set theory is a theory about vagueness, uncertainty enabling one to use nonprecise, ill-defined concepts and yet work with them in a mathematically strict sense [1]. Automatic Control theory has developed in the last decades from an emperically oriented technique into a strongly mathematically orientated rigid technique, requiring precision, well defined concepts and exact data. Nevertheless vagueness and subjectivity still play a role as is pointed out further below.

The introduction of the stability investigation approach by means of frequency diagrams (Nyquist, Nichols, Bode) created an exact method, the design criteria however remain vague and subjective. No definitive answer can be given as to what gain and phase margin, maximum relative error etc. have to be chosen to achieve a "good" system performance. The big spread of these criteria to be found with several authors, dependant on their personal views and experience is thus not surprising. Hence the introduction of different criteria like that of Ziegler and Nichols. The root locus method of Evans suffers from this same ambiguity as no exact values for the damping factors exist. The introduction of the integral error criteria was a step forwards in the exact determination of the optimal system, but in fact the vagueness here has been shifted to the choice of a particular criterion. The use of more complex performance criteria enables the incorporation of several desired factors in the optimisation. The decision as to which factors have to be accounted for and to what extent, is still subjective. Thus, notwithstanding the creation of numerous mathematical control techniques, the final decision about the "goodness" of a system's behaviour remains a personal, subjective task. Under the surface of modern control techniques subjectivity, vagueness - perhabs unconsciously - still does play a role. Furthermore, in non-engeneering systems, the so-called "soft systems" subjective matters are almost predominant. A theory of vagueness would be very useful here, to put it mildly[2].

Apart from this kind of general rationale of the incorpora tion of vagueness in system's design, there is a much more practical reason for the particular kind of fuzzy control system used in this research. Complex industrial plants such as chemeical reaction processes often are difficult to control automatically. In some cases plant models can be derived from the underlying physical or chemical properties of the process, but it is well known that this requires very complicated calculations, and that even under various approximations the final model often is very difficult, of high order, non-linear, time vatying etc. The method of parameter estimation to obtain a purely mathematically described behavioristic model may also require a very elaborate computation. When nonlinearity, time variance and stochastic disturbances have an important effect, modelling methods become still more complicated. Control theory however relies on modelling as an important step in the design process.

However it is interesting to note that in many cases the control of a process by a human operator is far more succesful than any such automatic control. Hence it seems useful to investigate the control policy of the operator. As the strategy he uses is vague and qualitatively described, the use of fuzzy set theory in such an investigation is self evident. This was also the rationale behind the "fuzzy logic controller" recently reported by Mamdani and Assilian [3]. In what was the first real control application of fuzzy set theory, they achieved a succesful control of a small boiler-steam engine combination, better even than a conventional DDC controller. The present work follows the same idea of using fuzzy rules as a control algorithm.

A warm water plant which had difficult control properties such as nonlinearity and variability, has been controlled by a fuzzy algorithm based on the experience of a human operator. From a set of linguistic rules which describe the operator's control strategy a control algorithm is constructed where fuzzy sets define the words used. Several types of such an algorithm are implemented and compared with each other, in behaviour as well as in structure. An alternative algorithm - mathematically equivalent to the other - is proposed to speed up the computation [4].

2 The Fuzzy Linguistic Control

The development of the theory of fuzzy sets and algorithms [5] makes it possible to build a control algorithm based on a very common kind of inexact information, namely information expressed in natural language. This linguistic information may be obtained from an experienced human process operator. This is done by demanding a qualitative description in his own words of the control strategy he uses and how he reacts in a situation. Thus the operator may be able to express his control strategy as a set of linguistic decision rules of the form:

if "increase in temperature is big" then "decrease pressure a lot" , else, if "increase in temperature is low" then "decrease pressure a little" , else, etc.

Clearly such expressions can be described as fuzzy sets on the universes of discourse "increase in temperature" and "decrease of pressure", respectively. Thus by defining the appropriate fuzzy sets and translating the rules as fuzzy implications of the form: if A then B, as functions of those fuzzy sets (A and B), the human control strategy can be converted into a control algorithm and implemented on a computer as outlined below. (In the appendix the precise mathematical derivation of the fuzzy control algorithm is presented. Here a less formal outline of the method will be given.)

Basic to the whole approach is the fuzzy implication (rule)

if A then B

where A and B are fuzzy sets, like "high temperature", "small pressure", on the universes of discourse input and output respectively. Considering this rule as a kind of equivalent of a system mapping the next question is: what will the output be to a certain input A' ? In other words, given the rule: if A then B, and the input A', what will be the output B' ? An expression for this is derived using the compositional rule of inference [5] in the appendix.

The next stage is the observation that the control algorithm clearly is composed of several rules; in different situations the human operator will apply different actions. The algorithm will have a form like

if A_1 the B_1 , else , if A_2 then B_2 , else .....

This set of rules will be evaluated by identifying the "else" connective as the union operator between fuzzy sets. The rules can be evaluated seperately and the results are combined using the max operator. Thus given a certain input A' resulting in an output of the first rule: B_1' , of the second rule: B_2' , etc., the resulting overall fuzzy output B' will be:

$$B' = \max (B_1' , B_2' , \ldots\ldots)$$

The extension of this single-input-single-output type to a more complex form of system having e.g. two inputs and one output with rules like

if A then (if B then C)

is a straightforward one. The same approach still applies (see appendix).

In the particular kind of application of this system concept to a process controller the input to the controller - temperature error - and the output of the controller - process input: flow - were both non fuzzy, deterministic values. The approach to cope with a non fuzzy input is explained in the appendix in two different ways and is quite straigtforward. The result of evaluating the fuzzy algorithm for a particular deterministic input is still a fuzzy output set ranging over the whole possible set of outputs. In order to obtain one deterministic output value from this fuzzy output set a decision procedure has to be adopted to make a choice as to which particular (non fuzzy) value is a good representative of the fuzzy set. The decision procedure applied here is to take that output value at which the membership function is maximal (see appendix).

2.1 The Process

This fuzzy system concept has been applied to design a controller for the temperature of a warm water plant, built on a laboratory scale. Figure 1 shows a schematic diagram of the plant. the warm water tank is divided into several compartments. The cold water stream enters the tank with a variable flow F2, passes the compartments in sequence and leaves the tank in the last compartment. This water is heated by a heat exchange unit in which hot water (at about 90^{o} centigrade) flows with a variable flow F1. The aim is to control the temperature of the water in one of the compartments for different temperatures and steady state values of the flow F2 by adjusting the dynamic values of F1 and F2. In this application the temperature of the water leaving the heating compartment has been controlled to minimize time delay problems. Usually a constant amount of liquid (i.e. water) of a certain temperature is required from the process, so the flow F2 has to be kept constant during steady state. Only during a change to another desired temperature the flow F2 can be changed, the main control variable however, is flow F1 of the hot water.

Earlier investigations of the process had shown that this process had difficult control properties, arising out of non-linearities, assymetric behaviour for heating and cooling, noise and dead time. Also the ambient temperature influences the process behaviour. To get a comparative idea of the performance of the fuzzy controllers an ordinary PI-controller has been implemented as well. This PI-controller has been optimally adjusted for an experimentally fitted model consisting of two equal time constants and time delay (time delay = 10 sec, time constants = 40 sec). The optimal values of the integral gain K_I and the proportional gain K_P for three different integral error criteria, the ITAE, IAE and the ISE, of this digital PI-controller are shown in table 1.

	ITAE	ISE	IAE
K_I	0.018	0.019	0.020
K_P	1.35	3.02	1.94

TABLE 1 Optimal K_I and K_P values for a digital PI-controller (sample time 1 sec)

One of the main difficulties of this controller was its need of adjustments to operate over a wide range of desired temperatures. It is clear that a more sophisticated controller (stochastic model, adaptive) than just a PI type would have a better performance. Hence the comparison between the PI control and "fuzzy control" should be regarded as only a rough indication of relative performance.

2.2 The Algorithm

The described fuzzy controller resulted in the following algorithm:

Every rule i associates a fuzzy flow (fl) subset to a fuzzy temperature (t) subset, represented by their membership functions:

$$\mu_i(t) \rightarrow \nu_i(fl) \qquad i = 1, 2, 3, \ldots, I$$

The actual action applied, fl_0, can be computed from the measured temperature t_0 as follows.

The membership values at the temperature t_0 are determined for each rule

$$\mu_1(t_0), \mu_2(t_0), \ldots, \mu_I(t_0)$$

The implied fuzzy subsets for the flow fl have a membership function λ that can be calculated for each rule as

$$\lambda_i(fl) = \min [\mu_i(t_0) \; ; \; \nu_i(fl)] \qquad i = 1, 2, \ldots, I$$

The overall fuzzy subset for flow is obtained by using the "or" statement

$$\lambda(fl) = \max_i \min [\mu_i(t_O) ; \nu_i(fl)] \qquad i = 1, 2, \ldots, I$$

The result is a fuzzy subset which ranges over all values of the flow. As the action is taken at the maximum value of the membership function of this fuzzy subset, it can be determined directly by taking that value of the flow fl_O, for which the following holds

$$\lambda(fl_O) = \max_{fl} \max_i \min [\mu_i(t_O) ; \nu_i(fl)] \qquad i = 1, 2, \ldots, I$$

3 The Fuzzy Controllers

3.1 The Fuzzy Sets

The fuzzy sets used in this application were of a continuous form. An uniform structure of the membership function for all fuzzy sets was chosen, namely the continuous function

$$\mu(x) = (1 + (a(x-c))^b)^{-1}$$

(see figure 2). This choice has the advantage that the desired shape of the fuzzy set can be adapted by just three parameters : c alters the point of minimum fuzziness ($\mu=1$), a the spread and b the contrast. Because the decision procedure would become too time-consuming in the continuous case, the fuzzy output sets were calculated at finite quantized intervals of the support set (flow). The definitions of the fuzzy sets used are shown in table 2. F1 is quantized in 12 levels, dF1 in 15 and F2 in 18.

x: temperature error, dx: change in error, F1: warm water flow, dF1: change in F1, F2: cold water flow.

NAME	SUPPORT SET	MEMBERSHIP FUNCTION
not small	x	$1 - (1+0.5x)^{-1}$
small	x	$(1+0.5x)^{-1}$
very small	x	$(1+x^4)^{-1}$
slightly small	x	$(1+0.5x)^{-1}$ for $x \geqslant 1$ else 0.
small	x	$(1+(3(x-1))^2)^{-1}$
medium small	x	$(1+(3(x-0.5))^2)^{-1}$
extremely small	x	$(1+(3x)^2)^{-1}$
small	dx	$(1+(3dx)^2)^{-1}$
medium	dx	$(1+(3(dx-0.5))^2)^{-1}$
big	dx	$(1+(dx-2)^2)^{-1}$
very big	F1	$(1+2(F1-12)^2)^{-1}$
very small	F1	$(1+2(F1)^2)^{-1}$
near st.state	F1'	$(1+(3(F1'-1))^2)^{-1}$
very near st.state	F1'	$(1+(3(F1'-0.5))^2)^{-1}$
small	dF1	$(1+(2(dF1-0.2))^2)^{-1}$
medium	dF1	$(1+(2(dF1-1))^2)^{-1}$
big	dF1	$(1+(dF1-3)^2)^{-1}$
very big	F2	$(1+2(F2-18)^2)^{-1}$
very small	F2	$(1+2(F2-1)^2)^{-1}$

TABLE 2 Definitions of the Fuzzy Sets used

3.2 Heuristic Structure

Whereas in [3] just one fuzzy control algorithm has been applied to a real dynamic process with succes, in this research three types of such fuzzy algorithms have been tested. In stead of asserting one fixed structure of the human operator's control heuristics, namely that a process operator generally uses error and rate of change of error to calculate a change in the value of the process input, several different heuristics have been applied. The reason for this was the fact that one part of the control - keeping the temperature accurately at a desired value - appeared to be difficult for a human controller. It was extremely difficult to avoid oscillations around the setpoint. Hence three strategies for this "steady state" control have been tested:

(1) the operator uses error and rate of change of error to affect a change of flow (process input).

(2) the operator only uses the error as information and compensates by changing the flow.

(3) the operator uses error and adjusts the flow above or below neutral position.

In this third strategy the controller was supposed to know what absolute value of the flow (F1) was the steady state position, hence a static flow-temperature characteristic was assumed to be known. A summary of these three different strategies is given in table 3.

	Observation	Action
strategy 1	error and change in error	change in flow F1
strategy 2	error	change in flow F1
strategy 3	error	flow F1 plus static information

TABLE 3 Control Heuristics

Because the aim of the control was not only to keep the temperature accurately at a desired value, but also to perform step changes in temperature as fast as possible, the set point change strategy should obviously have a kind of bang-bang character, both for flows F1 and F2 (the latter is only used during this change as stated earlier).

3.2 The Rules

The first strategy resulted in the following set of rules

if x "not small" then F1 "very big"
then F2 "very small"
if x "small" then F1 "very small"
then F2 at steady state
if x "very small" then F2 at steady state
then if increase of x "small" then decrase of F1 "small"
then if increase of x"medium" then decrease of F1 "medium"
then if increase of x "big" then decrease of F1 "big"

These are the five rules to control a temperature below setpoint while it is increasing. Apart from the second rule a completely symmetric set of rules was applied in the other cases.

The second strategy was realized by the following rules

if x "not small"	then	F1 "very big"
	then	F2 "very small"
if x "slightly small"	then	F1 "very small"
	then	F2 at steady state
if x "small"	then	increase of F1 "big"
	then	F2 at steady state
if x "medium small"	then	increase of F1 "medium"
	then	F2 at staedy state
if x "extremely small"	then	increase of F1 "small"
	then	F2 at steady state

The additional refinement of the "small" region required an appropriate modification of the previous fuzzy set "small" (see table 2).

The third strategy which has been applied consisted of the following set of rules

if x "not small"	then	F1 "very big"
	then	F2 "very small"
if x "small"	then	F1 "near steady state"
	then	F2 at steady state
if x "very small"	then	F1 "very near staedy state"
	then	F2 at steady state

Because the static flow-temperature characteristic was very sensitive to the environment, the algorithm was set up to enable alterations of this characteristic during running time.

3.4 Results

The overall results of these three types of controllers have been summarized in table 4 and compared with a PI type controller mentioned above. In view of the bang-bang rules it is not

Controller	Rise Time (minute)	Overshoot (centigrade)	Temp.Variations (centigrade)
classical PI type	0.7 min	1.5^o	0.4^o
first fuzzy type	0.3 min	less than var.	1.5^o
second fuzzy type	0.3 min	"	1.5^o
third fuzzy type	0.3 min	"	0.5^o

TABLE 4 Performance of Different Controllers on a Step Response of 10 degrees centigrade.

surprising that the systems with the fuzzy controllers all show much faster step responses than the classical PI type control system (for a step of 10^o centigrade about 0.3 minute against 1.5 minute for the PI controller). However the first two controllers behaved like the human operator in that their accuracy was poor (1.5^o centigrade oscillations around the setpoint against 0.4^o for the PI controller). The warm water process with the third type fuzzy controller showed the best performance. It combined the same high speed step response as the other fuzzy controllers (0.3 minute) with the same accuracy as that of the PI controller (0.5^o variations).

3.5 Discussion

A possible explanation of the better results of the last fuzzy controller type could be the additional information about the "neutral" steady state flow position. The introduction of this steady state information has the disadvantage that the controller has to be readjusted for each different desired temperature value. The sensitivity of these settings to changing surroundings is another problem. The fact that the actual readjustment of these settings during running time was performed by the human operator indicates that a vague guess of this steady state flow value might be sufficient. However it has to be realized that in some industrial processes even a guess of such steady state characteristics may be impossible.

Another highly intuitive way of explaining the differences in behaviour of these three fuzzy controllers could be to relate their structure to those of conventional controllers. Looking only at the "steady state" rules, it can be observed that the inputs and output of the first type fuzzy controller are similar to those of a PI type incremental control algorithm. The input-output quantities of the second type are those of a purely I type incremental algorithm and finally the third type has an input and output identical to those of a P type controller using a positional algorithm (see table 3). It should be emphasized that this supposed analogy lacks any rigid basis. The sort of combined bang-bang and "PI" nature makes an explanation of the results from only this second point of view even more doubtful. Clearly more detailed study on such an analogy should therefore be done (and is currently being conducted) before its conclusions are used to assess the accuracy and stablity.

One observation which can definitively be made is that this kind of fuzzy control is very well suited for an easy implementation of a time optimal control. The calculation of a switching line for the bang-bang control of a noisy time delay system is extremely difficult and the simplicity of this fuzzy bang-bang control is therefore an important advantage.

3.6 Further Remark

It is possible to speed up this fuzzy algorithm by using an alternative approach to decide at the beginning to which fuzzy temperature subset the measurement belongs. This is interpreted as to mean the fuzzy subset where the measured point has the highest membership grade. This decision gives thus the rule number (i_O) at which

$$\mu_{i_O}(t_O) = \max_{i} \mu_i(t_O)$$

Having determined this rule number, the appropriate calculations are carried out for this rule only. The action is then taken at that flow fl_O at which

$$\lambda(fl_O) = \max_{fl} \min [\mu_{i_O}(t_O) ; \nu_{i_O}(fl)]$$

This method not only saves a considerable amount of computing time but also has a kind of intuitive appeal left. Its equi-

4 Conclusions

A comparison has been made between the response of the system for the three different fuzzy controllers and with DDC controllers of a non fuzzy nature. The DDC controllers had a PI action; the setting of this action was optimised according to the ISE-, IAE- and ITAE-criteria on a linearised model.

All the fuzzy controllers showed a faster step response of the system than was possible with the DDC-controllers. However, it was more difficult to get accurate control of the temperature (see table 4). The simplest fuzzy controller, the third type, showed the best performance and combined a high speed response with the same accuracy as that of the optimal DDC-controller. The other two fuzzy controllers showed a tendency to oscillations around the steady state value.

It has been shown that the three different types of fuzzy controllers show some similarities with proportional nal and integral actions. Although the results of this preliminary research on fuzzy control are promising, the accuracy and stability problem needs to be inverstigated more deeply. This kind of fuzzy control is essentially nonlinear but it is the way the particular control algorithm is derived which is the novelty and major contribution of this method based on fuzzy set theory. The easy way of implementing the experience of a human operator in the controller makes the application of fuzzy linguistic rules attractive for those processes that are already controlled by operators. This is particularly true in cases where automatic control following the usual methods requires time consuming and complex modelling and control methods.

Acknowledgement

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5 Appendix : Fuzzy Systems

A fuzzy subset A of a universe of discourse (support set) X is characterised by a membership function $\mu_A(x)$. This function assigns to each element $x \in X$ a number $\mu_A(x)$ in the closed interval $[0,1]$, which represents the grade of membership of x in A [5]. Three basic operators used in fuzzy set theory are defined as follows:

(a) The union of the fuzzy subsets A and B of the universe of discourse X is a fuzzy subset, denoted $A \cup B$, with a membership function defined by

$$\mu_{A \cup B}(x) = \max [\mu_A(x) \; ; \; \mu_B(x)] \qquad x \in X$$

The union corresponds to the connective "OR".

(b) The intersection of the fuzzy subsets A and B is a fuzzy subset, denoted $A \cap B$, with a membership function defined by

$$\mu_{A \cap B}(x) = \min [\mu_A(x) \; ; \; \mu_B(x)] \qquad x \in X$$

The intersection corresponds to the connective "AND".

(c) The complement of a fuzzy subset A is a fuzzy subset, denoted $\neg A$, with a membership function defined by

$$\mu_{\neg A}(x) = 1 - \mu_A(x) \qquad x \in X$$

Complementation corresponds to negation "NOT".

The definition of a fuzzy set enables us to deal with the information contained in the experience of a human operator.

For instance linguistic expressions, such as the flow is "big", "medium", "small", "not big", etc. clearly are fuzzy subsets of the universe of discourse "flow".

Furthermore to represent in a fuzzy way the concept of a system mapping from an input to an output set, the concept of a fuzzy conditional statement (implication) is introduced. The system is described as a set of fuzzy conditional statements of the form

if "input is big" then "output is medium"

The membership function corresponding to a fuzzy conditional statement S: if A then B, given the fuzzy subset A of the universe of discourse X and the fuzzy subset B of Y, is defined by [5]

$$\mu_S(y,x) = \min\left[\mu_A(x)\ ;\ \mu_B(y)\right] \qquad x \in X\ ,\ y \in Y \qquad (1)$$

The complete system is described by a set of such fuzzy implications* e.g.

if "input is big" then "output is medium"
or (else)
if "input is medium" then "output is small"

Using the above metioned definition of the "or" connective the final fuzzy implication S composed of two implications: if A_1 then B_1 or (else) if A_2 then B_2 , has the membership function

$$\mu_S(y,x) = \max\left[\min[\mu_{A_1}(x); \mu_{B_1}(y)] ; \min[\mu_{A_2}(x); \mu_{B_2}(y)]\right] \quad (2)$$

This can be extended to the case of more than two rule statements.

Having defined the relation between fuzzy subsets, the next step is to calculate the infered fuzzy subset, given a certain implicand fuzzy subset. Knowing the rule: if "input is big" then "output is medium" the question arises what will be the output when the "input is very big"? For this, the following compositional rule of inference is used. Given a fuzzy implication S: if A then B, the fuzzy subset B', infered from a given fuzzy input set A' (A and A' fuzzy subsets of X, B and B' of Y], has a membership function defined by [5]

$$\mu_{B'}(y) = \max_x \min\left[\mu_{A'}(x) \ ; \ \mu_S(y,x)\right] \quad (3)$$

The input to the system in this control application was considered to be precise, not fuzzy. There is no fuzzy input, hence there is no need to apply the compositional rule of inference. Using the intuitive meaning of a fuzzy implication: if A then B, the implied output can never achieve a higher degree of truth than that of the implying input. That would be contrary to the nature of an implication. Hence one obtains the fuzzy output B up to the degree of membership of the measured value x_0 in the fuzzy input A. This gives the fuzzy output set

$$\mu_{B'}(y) = \min \left[\mu_A(x_0) \; ; \; \mu_B(y) \right] \quad = \quad \mu_S(y, x_0)$$

An alternative way to obtain the same result is to interpret this input x_0 as a "fuzzy" input set A' with all membership values $\mu_{A'}(x)$ equal to zero, except the value at the measured point $\mu_{A'}(x_0)$ which is equal to one. Equation (3) - the compositional rule of inference - reduces then to

$$\mu_{B'}(y) = \mu_S(y, x_0)$$

The representation of a fuzzy system is used as an algorithm for a fuzzy controller: a decision has to be made as to which particular action should be taken and fed into the process. The decision procedure applied here is to take that value y_0 at which the final membership function is a maximum, that is y_0 at which

$$\mu_{B'}(y_0) = \max_y \mu_{B'}(y) = \max_y \max_x \min \left[\mu_{A'}(x) ; \mu_S(y,x) \right] \qquad (4)$$

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Footnote page 21

* the extension to the case of an implication of the form: if A the (if B then C), is straightforward:

$$\min[\mu_A(x) \; ; \; \min[\mu_B(y) \; ; \; \mu_C(z)]] = \min[\mu_A(x);\mu_B(y);\mu_C(z)]$$

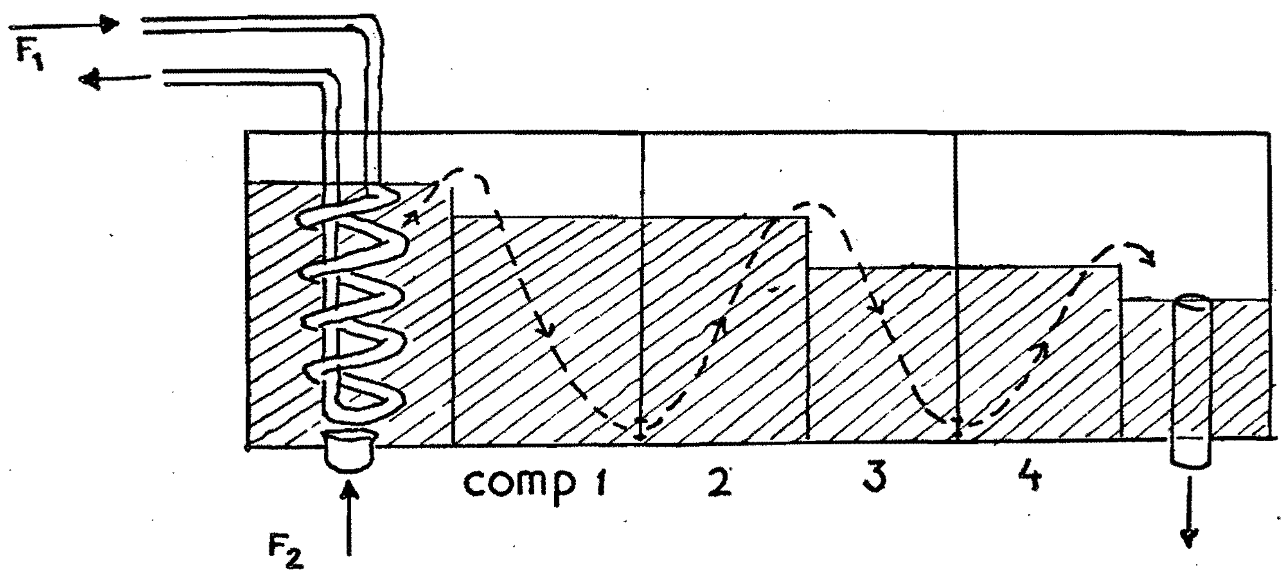

figure 1 : schematic diagram of the plant

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THE APPLICATION OF FUZZY CONTROL SYSTEMS TO INDUSTRIAL PROCESSES

P.J. King and E.H. Mamdani

INTRODUCTION

Complex industrial processes such as batch chemical reactors, blast furnaces, cement kilns and basic oxygen steelmaking are difficult to control automatically. This difficulty is due to their non-linear, time varying behaviour and the poor quality of available measurements. In such cases automatic control is applied to those subsidiary variables which can be measured and controlled, for example temperatures, pressures and flows. The overall process control objectives, such as the quality and quantity of product produced, has in the past been left in the hands of the human operator.

In some modern plants with process control computers, plant models have been used to calculate the required controller settings automating the higher level control functions. The plant models whether they are based on physical and chemical relationships or parameter estimation methods are approximations to the real process and may require a large amount of computer time. Some successful applications have been reported, but difficulties have been experienced where processes operate over a wide range of conditions and suffer from stochastic disturbances.

An alternative approach to the control of complex processes is to investigate the control strategies employed by the human operator. In many cases the process operator can control a complex process more effectively than an automatic system; when he experiences difficulty this can often be attributed to the rate or manner of information display or the depth to which he may evaluate decisions.

The operator usually expresses his control strategy linguistically as a set of heuristic decision rules. It is difficult to convert this qualitative control strategy into a quantitative controller design due to the imprecise nature of the rules. Therefore means of implementing the human operators control rules directly as an automatic control system is of interest. Zadeh's development of fuzzy sets[1] and fuzzy algorithms[2] provides a means of expressing linguistic rules in a form suitable for processing using a computer. In this paper are reported some case studies on pilot-scale processes in which heuristic strategies using fuzzy statements are applied to the control of dynamic processes.

THE CONTROL SYSTEM

The structure of the control system is shown in Fig. 1; the heuristic decision rules replace a conventional feedback controller in the error channel. The calculation of the control action is composed of the following four stages:

1) Calculate the present error.
2) Assign the error value to a fuzzy variable such as Positive Big.
3) Evaluate the decision rules using the compositional rule of inference.
4) Calculate the determimistic input required to regulate the process.

The exact form of the decision rules and the variables used in them will depend on the process under control and the heuristics employed. In general the process operator uses error (E) and rate of change of error (CE) to calculate a change in the value of the process input (CU) and the decision rules are designed to have the same effect. This approach also corresponds to the versatile proportional + integral controller used frequently in the process industry.

The error value and the change of error values calculated are quantised into a number of points corresponding to the elements of a universe of discourse, and the values are then assigned as grades of membership in seven fuzzy subsets as follows: 1) PB = positive big, 2) PM = positive medium, 3) PS = positive small, 4) PO = positive nil, 5) NO = negative nil, 6) NS = negative small, 7) NM = negative medium and 8) NB = negative big. The relationship between measured error or change in error value and grade of membership are defined by look-up tables of the form given in Table 1. These basic subsets may then be used with the three basic operators of union, intersection and complement to compute such values as "Not positive big or medium". Hedges may also be used but to avoid complications these were not implemented in this study.

The decision rules are implemented as a set of fuzzy conditional statements of the form,

"If E is NB then CU is PB".

This expression is evaluated using the compositional rule of inference for a particular value of error E as described by Zadeh[3]. The result is a value for change of input CU for any given value of error E. In most cases the rules are more complex than the example above and for the system using change of error (CE) and error (E) will be of the form,

"If E is PB or PM then if CE is NS then CU is NM"

but the same methods of evaluation still apply.

Several rules are required to completely define a control system; the results of evaluating each rule are combined using the union operator (max) to give an overall fuzzy value for the control action. For example,

"If A_1 then (if B_1 then C_1)

Or If A_2 then (if B_2 then C_2)"

etc.

So given values of measurements A_1', A_2', B_1', B_2' etc the individual rules results will be C_1', C_2' etc and these are combined to give the overall resulting control action,

$$C' = \max (C_1', C_2' \text{ etc})$$

Hence more than one rule may contribute to the computation of a control action.

The result of evaluating the fuzzy rules for a particular set of input values is a fuzzy set of grades of membership for all possible control actions. In order to take a deterministic action one of these values must be chosen, the choice procedure depending on the grades of membership and the particular application. In this work the control value with the largest grade of membership was selected, except in the cases where several control actions had the same (largest) grade of membership. In these cases where more than one peak or a flat peak is obtained the value midway between the two peaks or in the centre of the plateau was selected. Typical results are shown in Fig. 2 as curves of grade of membership versus control action. The shape of these curves can be used to assess the quality of the control rules used; Fig. 2 (a) shows a single strong peak indicating one dominant control rule in this region. Fig. 2 (c) shows a fuzzy result which indicates an absence of a good set of rules, while Fig. 2 (b) with two peaks shows that at least two strong and contradictory rules are present. In both these latter cases some modification of the control rules may be necessary to obtain good control.

The rules are evaluated at regular intervals in the same way as a conventional digital control system. The choice of sampling interval depends on the process being controlled and should be selected so that at least five significant control actions are made during the process settling time.

Control of a Boiler and Steam Engine

Mamdani and Assilian[4,5] conducted an application study on a small boiler steam engine combination. The heat input to the boiler was used to control the boiler pressure and the steam engine speed was controlled by adjusting the throttle opening at the input of the engine cylinder.

The process dynamics could be approximated by two first order lags in series, with time constants and gains varying depending on the operating conditions. The rules were evaluated with a 10 second sampling interval.

Operating experience and technical knowledge of the process was used to specify two sets of heuristic fuzzy control rules for the two feedback loops. The look up tables used to specify the fuzzy subsets for values are all similar in form to Table 1; the number of quantisation levels used was 14 for error, 13 for change of error, 15 for heat input change and 5 for throttle input change. Details of the rules used are given in the Appendix.

The fuzzy control rules were implemented on a PDP-8 computer and used to control the plant and a conventional digital controller was also used to control the two loops. The control results obtained for the pressure loop are shown in Fig. 3; the digital controllers were difficult to adjust as the process is highly non-linear and good control could not be achieved at different operating conditions with the same controller settings. The fuzzy control system was much less sensitive to process parameter changes and gave good control at all operating points, in many cases better than the conventional control system results. This can largely be attributed to the non-linear nature of the heuristic rules, which could be used to give a rapid response and a small amount of overshoot.

Temperature Control of a Stirred Tank

The results obtained using fuzzy control of the steam engine were much better than expected, so a second application study on the temperature control of a stirred vessel, part of a batch reactor process, is currently being conducted. The process consists of an 80-gallon stirred tank which can be heated by a steam heating coil and is cooled by recirculating tank fluid through an external heat exchanger. The vessel temperature is controlled by changing the steam or recirculating flows.

The dynamics of the stirred tank can be approximated for small changes in temperature by an integrator with different gains for heating and cooling and a pure time delay of 1 minute for heating and 0.6 minutes for cooling. Initial control experiments were conducted using the pressure control rules developed on the steam engine, while the process dynamics differ considerably it was thought that similar heuristic rules would be effective, if the values used to define the fuzzy subsets of magnitude were suitably adjusted.

The results obtained by simulating these control rules in the simple process model are shown in Fig. 4 for set point changes. Two different sampling intervals were used to evaluate the rules, 30 seconds and 1 minute; in both cases when delays were included the temperature oscillated about the desired value. If the delay was removed from the process model, however, good control responses were obtained. In the case with a one minute sampling interval the process finally settled to a steady value; this result was confirmed in practice as the results in Fig. 5 show the response of the process with a 1-minute sampling time interval to set point changes.

The input quantisation levels for the fuzzy subsets of value can be adjusted to improve the system response. However, more detailed simulations over a wide range of quantization values show that the delay is the cause of the instability. The steam engine had negligible pure time delays and the rules formulated using error and change of error values were adequate. However, when delays are present the rules must also account for the control inputs applied to the process which have not yet been observed as a change in the process output. For systems with pure time delays the control rules will have to include a fuzzy model of the system to predict the future output of the system, or in other words previous values of control input will have to be included in the rules. New rules to control the whole of the reactor process are being formulated along these lines, but no results are available yet.

CONCLUSIONS

The results obtained so far show that processes can be controlled effectively using heuristic rules based on fuzzy statements. To obtain good control the fuzzy rules must be correctly formulated to take account of time delays when they occur; this conclusion is similar to that arrived at for conventional controllers when delays are present.

The designer requires some knowledge of the process in formulating the rules, for instance knowledge of process delays and speed and magnitude of response, but only approximate values are required and can usually be obtained by operating the process. The fuzzy control system described is inherently non-linear and phase plane plots showing the system quantization levels, the rules used in each area and the magnitude of control action have been used to advantage as a design aid.

The approach described here is not proposed as an alternative to conventional control in situations where this is effective; however, in complex systems the use of fuzzy algorithms for control may be a non-trivial alternative approach. The fuzzy measurements required can be obtained from instruments or directly from the human operator, and there is considerable scope for interaction between the man and machine due to the fuzzy nature of the algorithms.

In the most complex situations, control is often based not on one variable but on a combination of variables or events which the human operator recognises as significant. Before control rules can be formulated these patterns must be identified as significant measured variables. While the approach described does not resolve this problem the use of fuzzy concepts will be of use in this situation.

TABLE 1. - Look-up Table Relating Fuzzy Subsets to Quantized Error Values

	-6	-5	-4	-3	-2	-1	-0	+0	+1	+2	+3	+4	+5	+6
PB	0	0	0	0	0	0	0	0	0	0	0.1	0.4	0.8	1.0
PM	0	0	0	0	0	0	0	0	0	0.2	0.7	1.0	0.7	0.2
PS	0	0	0	0	0	0	0	0.3	0.8	1.0	0.5	0.1	0	0
PO	0	0	0	0	0	0	0	1.0	0.6	0.1	0	0	0	0
NO	0	0	0	0	0.1	0.6	1.0	0	0	0	0	0	0	0
NS	0	0	0.1	0.5	1.0	0.8	0.3	0	0	0	0	0	0	0
NM	0.2	0.7	1.0	0.7	0.2	0	0	0	0	0	0	0	0	0
NB	1.0	0.8	0.4	0.1	0	0	0	0	0	0	0	0	0	0

APPENDIX

Steam Engine Control Rules

Pressure Control Algorithm

Pressure Error = PE, Change in Pressure Error = CPE and heat input change = HC.

If PE = NB then if CPE = not (NB or NB) then HC = PB

Or

If PE = (NB or NM) then if CPE = NS then HC = PM

Or

If PE = NS then if CPE = PS or NO then HC = PM

Or

If PE = NO then if CPE = (PB or PM) then HC = PM

Or

If PE = NO then if CPE = (NB or NM) then HC = NM

Or

If PE = PO or NO then if CPE = NO then HC = NO

Or

If PE = PO then if CPE = (NB or NM) then HC = PM

Or

If PE = PO then if CPE = (PB or PM) then HC = NM

Or

If PE = PS then if CPE = (PS or NO) then HC = NM

Or

If PE = (PB or PM) then if CPE = NS then HC = NM

Or

If PE = PB then if CPE = not (NB or NM) then HC = NB

Or

If PE = NO then if CPE = PS then HC = PS

Or

If PE = NO then if CPE = NS then HC = NS

Or

If PE = PO then if CPE = NS then HC = PS

Or

If PE = PO then if CPE = PS then HC = NS

Speed Control Algorithm

Speed Error = SE, change in speed error = CSE and change in throttle opening = TC

If SE = NB then if CSE = not (NB or NM) then TC = PB

Or

If SE = NM then if CSE = (PB or PM or PS) then TC = PS

Or

If SE = NS then if CSE = PB or PM then TC = PS

Or

If SE = NO then if CSE = PB then TC = PS

Or

If SE = PO or NO then if CSE = (PS or NS or NO) then TC = NO

Or

If SE = PO then if CSE = PB then TC = NS

Or

If SE = PS then if CSE = PB or PM then TC = NS

Or

If SE = PM then if CSE = PB or PM or PS then TC = NS

Or

If SE = PB then if CSE = not (NB or NM) then TC = NB

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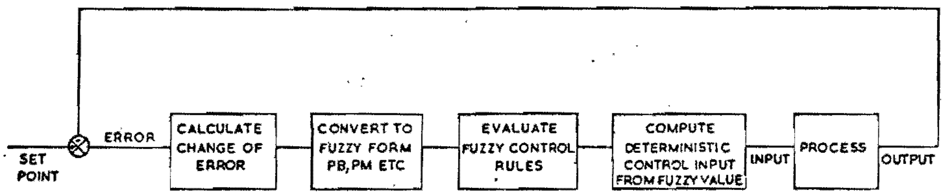

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FIG.1 CONTROL SYSTEM USING FUZZY RULES

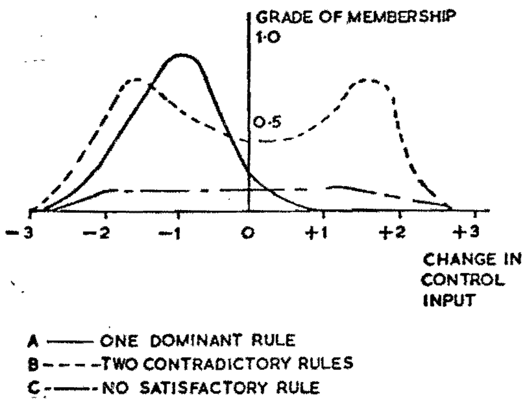

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FIG.2 CHANGE IN CONTROL INPUT CALCULATED BY FUZZY RULES

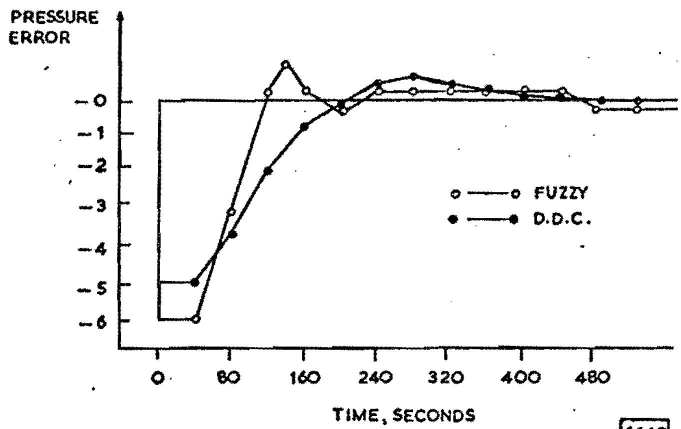

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FIG.3 PRESSURE CONTROL RESPONSES FOR D.D.C. AND FUZZY CONTROL SYSTEMS APPLIED TO A SMALL BOILER AND STEAM ENGINE

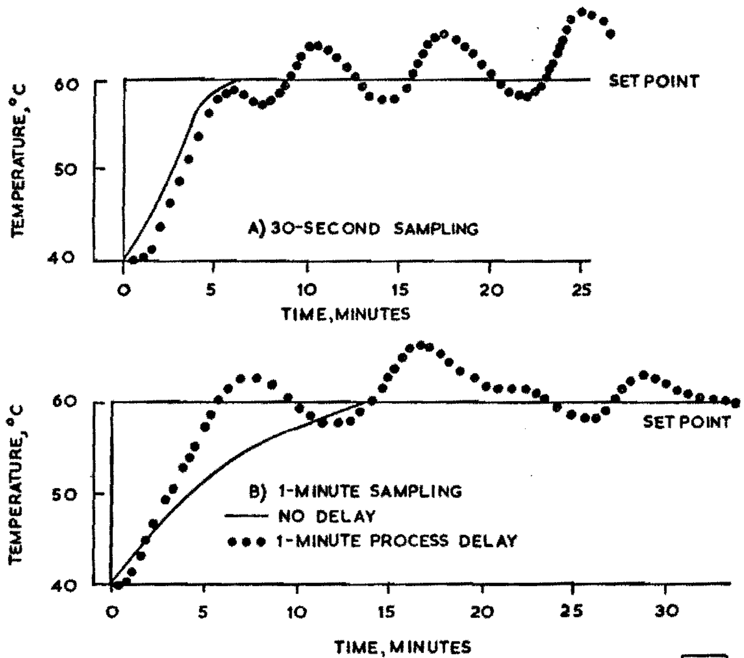

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FIG.4 SIMULATED RESPONSES OF STIRRED TANK TO SET POINT CHANGES WITH AND WITHOUT DELAY

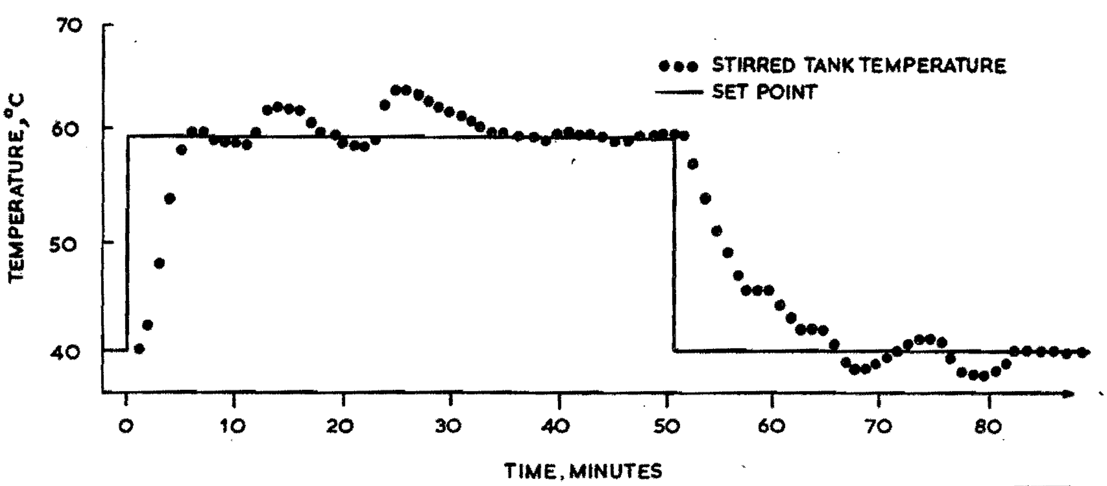

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FIG.5 RESPONSE OF STIRRED TANK TEMPERATURE TO SET POINT CHANGES WITH A 1-MINUTE SAMPLING INTERVAL

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Ladislav J. KOHOUT: TOPOLOGY AND AUTOMATA.

1. Introduction

Conventional topologies play an important role in the study of stability of continuous, dynamic and control, systems [1], [2]. Recently some attempts were made to unify automata and control theories [3], [4]. The topological methods have been usefully employed in these generalisations [5], [6].

It has been shown that, in certain contexts, standard topologies are too special to be applied to general systems [7], [8], or to automata [9] and that suitable generalised topologies are required [11].

Our present study has been motivated by the attempt to apply topological methods to studies of adaptivity, in particular, to the problems formulated by Gaines [10], . This note is an expanded and complete revised version of a previously unpublished note [13] that had some limited circulation earlier.

This note does not aim at the presentation of any non-trivial or new mathematical results. Its only aim is a rather elementary discussion of applications of topologies to automata theory and discussion of the semantics of some topological constructs in this context.

2. Intuitive Motivation

Set theoretical topological methods employ subsets or families of subsets of points and mappings or relations between them. We shall not work with individual states, inputs and outputs of an automaton but with subsets of them instead. Hence we shall aim at methods for description and manipulation of hyperstates, hyper-inputs and hyper-outputs.

In general, we are interested in reachability, controlability and stability of a state-determined system. For example, we want to determine a hyperstate that can be reached from a given hyperstate by the application of a particualr hyper-input etc.

Systems can be classified according to their dynamical properties, this classification forming a hierarchy of families of equivalence classes modulo a subset of hyperstates satisfying the condition that the mappings between the corresponding hyperstates of distinct systems are homeomorphic. This approach then naturally leads to use of continuous mappings, closures neighbourhoods, nets, etc. in generalised topologies.

The topological approach based on generalized topologies is by no means confined to crisp systems, for topological structures can be fuzzified. This leads in a natural way to generalised topologies "without points" (Koutský (1947, 1952), cf. Kohout (1975) p. 29 of 5.1).

Connection between various modal logics and their topological semantic models in the form of a Boolean algebra with an additional operator is well known (McKinsey (1941) McKinsey & Tarski (1948), Lemmon (1966). Usefulness of modalities for description of dynamics of systems has been pointed at in Kohout 1975, Gaines & Kohout 1975, Kohout & Gaines 1976. If topologies without points are used instead, this leads to a certain type of fuzzified modal logics.

3. Definition of Basic Concepts

A state-determined system is specified by the relation σ:

$\sigma: \quad I \times S \rightarrow S$

where S = {Si} is the set of states and I = $\{i_1, i_2, \ldots i_n, \ldots\}$ the set of inputs of the system.

The next state relation (NSR) σ can be decomposed into a family of relations Σ with regard to individual inputs:

$\Sigma = \{\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}, \ldots \sigma_{i_n}, \ldots\} \; i_1, i_2, \ldots i_n, \ldots \varepsilon I$

The mapping $\underline{u}$

$\underline{u}: \quad \mathcal{P}(A) \rightarrow \mathcal{P}(A)$

will be called a generalized closure operator [9] . The tuple $\langle A, \underline{u} \rangle$ will denote a general topology generated by the closure operator $\underline{u}$ on the carrier set A. Special structural properties of each topological space will be characterised by a set of axioms [9]. In particular, we shall be interested in axioms having a topological property [12]. This notion is closely connected with the concepts of hormeomorphism and continuous mapping.

Definition 1: (mapping continuous at a point ([12], p. 269-270).

Let f be a mapping of a closure space $\mathcal{R} = \langle R, \underline{u} \rangle$ into a closure space $\mathcal{S} = \langle S, \underline{v} \rangle$ The mapping f is said to be continuous at a point x of $\mathcal{R}$ if

$X \subset R, \; x \varepsilon \underline{u}(X)$ implies $f(X) \varepsilon \underline{v}(f(X))$.

The mapping is said to be continuous if it is continuous at each point x of R, or equivalently,

$X \subset A$ implies $f(\underline{u}(X)) \subset \underline{v}[f(X)]$

Definition 2: A homeomorphism is a bijective mapping f for closure spaces such that both f and f^{-1} are continuous.

Definition 3: A topological property is a property such that if a closure space $\mathcal{R} = \langle R,\underline{u}\rangle$ posseses this property then all homeomorphs of P also have this property.

In the sequal, an automaton will be viewed either as

a) a next state relation σ, or

b) a time system

a) Next state relations σ: IxS → S with inputs S and states I.

Each input i_k εI induces an input restriction σ_{i_k} on σ

$\sigma_{i_k}: \{i_k\} \times S \to S$

The projection Π of $i_k \times S$ into S is a family of subsets $\bar{S}_\Pi = \{S_{\Pi(i_1)}$ $S_{\Pi(i_2)}, \ldots S_{\Pi(i_k)}, \ldots S_{\Pi(i_k)} \cdots\}$ such that $\bar{S}_\Pi \subset P(S)$.

The elements of P(I) will also be called the input-base (i-base) generators. We can also define W- projection such that IXS_k is projected into I.

b) time system

P: T x S → S where T is the time order.

This T will induce a quasi order p on the set of states S. Note that if s_i precedes s_k. This does not exclude the situation where s_k also precedes s_i in another time interval. It is also possible that s_r never precedes itself. Hence P is transitive but neither reflexive nor symetric.

Obviously, p, Π, W will generate distinct generalised topologies with operators $\underline{TOP}_T$, $\underline{TOP}$, $\underline{TOP}_\Sigma$ respectively.

$\underline{TOP}_T$: $\mathcal{P}(S) \to \mathcal{P}(S)$

$\underline{TOP}_\Pi$: $\mathcal{P}(S) \to \mathcal{P}(S)$

$\underline{TOP}_W$: $\mathcal{P}(I) \to \mathcal{P}(I)$

Each relation will induce a corresponding generalized topology by means of the canonical expansion.

Definition 4: [12] p. 38 (Canonical expansion of relation to a class of sets A). For the definition see [12] p. 38.

4. Important closures

Topological properties and some important axioms for generalized topologies that appear in the literature were briefly reviewed in [9] All axioms used in the sequel can be found in the list given at pp. 26 - 27.

4.1 The Transition Closure $\underline{u}_\sigma$

This closure represents the set of possible states that are attained after a one-step transition.

$\underline{u}_\sigma(X) = X \cup \sigma(X),\ X \subset S$

It is obvious that this closure will be an OIA-topology. An OIA topology taken as a Boolean algebra with the operator $\underline{u}_\sigma$ is sometimes called an extension modal algebra.

Lemmon established the connection between extension algebras and T-modal systems. Hence the modal logic for description of transition closures can be based on a T-modal system [9]. . The modal operator in this system will determine the set of possible states after one step transition.

4.2 Accessibility closure

This closure is defined as the transitve closure of the next-state relation σ:

$\underline{\sigma}(X) = X \cup \sigma(X) \cup \sigma^2(X) \cup \sigma^3(X) \cup .. \cup \sigma^\mu(X) \cup ....$

This will be at least a pre-ordered (REFL + TRANS) OIMU-topology. In the case that the corresponding next-state closure also defines an A*-topology, the u will be an OIAU-topology.

It is apparent that the closures defined in this paragraph will be upper U-modifications (U: u(X)=u(u(X)) of the transition closures defined in 4.1.

On the other hand, accessibility closures must have properties induced by the quasi-order p defined in the section above, (they have to have the properties of TOP).

If we assume that an automaton can be at present in any single state s_i the topological space on the state space defined by accessibility closure will be a pre-ordered AIOU-topology. Hence it can be represented as a special case of S4 modal logic. It appears that time modal logics are relevant to description of segments of automata behaviour. Unfortunately, the state space itself is only a pre-ordered space hence even time modal logics of such generality as S4.2 are not sufficiently general to fully describe a general automaton. Also, the answer to the question of a single pre-finite extension of S4 which would describe all and only fine state machines is rather negative. There exist 5 pre-finite extensions of S4 each of which can describe some automata of a special type.

4.3 Generalised Measures

In some practical applications it may be more convenient to define generalised metrics first and then to find corresponding generalized topologies induced by the metric. For example, in the case discussed in the paragraph 4.2 we can define a pseudo-quasi-metric satisfying the following inequalities

(1) $d(x,\eta) \geqslant 0$

(2) $d(x,x) \geqslant 0$

(4) $d(x,z) \geqslant d(x,\eta) + d(\eta,z)$

where distance d is defined as inf. $\{d(x,\eta) - x\varepsilon X, y\varepsilon Y\}$

We can see that this cannot be quasi-metric, for we can leave simultaneously $X \leqslant \eta$ and $X \geqslant \eta$ with $x \neq y$

The first two conditions (1) (2) induce an OIM topology.

4.4. Iterations in generalized topologies

For investigation of limit cycles, it is useful to look at iteration. We take $g(X) = \sigma(X)$ and investigate iterations in this topology.

1. $g^{o}(X) = X$
2. $g^{\xi+1}(X) = g(g^{\xi}(X))$ for every ordinal.
3. $g^{\xi}(X) = \gamma$ exist. $g^{\xi}(X)$ $|0 \leqslant \gamma < \xi|$ for every limit ordinal .

We are interested for example in the smallest ordinal for which $g^{\xi+1} = g^{\xi}$.

5. Control-theoretical interpretation of some topological concepts.

Reachability and accessibility play an important role in the control theory and have importance in applied automata theory such as the theory of adaptive behaviour (Gaines, 1972) [10].

5.1 Accessibility

Apart from potential accessibility which can be interpreted in modal terms as possibility as discussed above in 4.2., some other forms of accessibility can be introduced. For example, the next-state accessibility is defined by the transition closure μ_σ. N-th step accessibility also has some practical importance. This can be defined by n-th iteration of the transition closure.

Attractors can be defined as the smallest family of accessibility closures of the set of all singletons of the state-space or equivalently as a U-modification of the next-state closure on the same set of singletons ($\underline{\mu}^{\xi}(x) = \underline{\mu}^{\xi}(x)^{+1}$).

5.2 Reachability

This can be defined as a binary predicate ... is reachable from ... .

Separability of two points or sets plays an important role here.

The usual H-separability **axiom and its** special instances ($\overline{H}$-, $\overline{\overline{H}}$-, $\mathcal{H}$-separibility) are not general enough for this application. Being used for standard topological spaces they are symmetrical, whereas the transition function in automata theory is not.

We propose that the following non-symetrical axioms should be studied:

G_o: $V_1(M) \cap N = 0$ (half of H-axiom)

G_1: $\frac{1}{2}$ of $\overline{H}$

G_2: $\frac{1}{2}$ of $\overline{\overline{H}}$ $\qquad [M, N \subset X;\ \underline{\mu}: \mathcal{P}(X) \to \mathcal{P}(X)]$

(V_1 is a neighbourhood of M - cf. 5.4 below) In the case that G_o holds, N is not accessible from M.

5.3 Interior

For a given topological space $\mathcal{S} = (S,\underline{\mu})$ and a subset $X \subseteq S$ interior is defined:

$int_u (X) = S - \underline{\mu}(S-X)$ when S-X is the complement of X.

Control-theoretical interpretation for the accessibility closure as defined in 5.2, then Fr_u (X) contains only the states accessible from both X and its complement.

6. ADAPTION AUTOMATA. (Gaines, 1972) [10].

6.1. Some important set operations. (Borůvka, 1976 : Introduction to the theory of groupoid & groups, 2nd Engl. ed, Birkhauser, Stuttgart)

6.1.1. We shall generalise some concepts of Borůvka.

Let S is a set, a $A \subset S$ and $P_i \subset \exp S$; $\bar{P}$ is a collection of P_i.

Operation $A \sqsubset \tilde{P}$:

$A \sqsubset \tilde{P} = \forall (\bar{P}_i | \bar{P}_i \between A, \bar{p}_i \subset \tilde{P})$ (the $\between$ has the meaning 'is incident with..').

Example: $\tilde{P}=((1,2,3),(2,3,4),(5,6))=(\bar{p}_1,\bar{p}_2,\bar{p}_3)$

$\bar{p}_1 \between \bar{p}_2$ but $\bar{p}_2 \not\between \bar{p}_3$ ($\bar{p}_2$ is not incident with $\bar{p}_3$).

Let $A=(1,4)$; then $A \sqsubset \tilde{P}=((1,2,3),(2,3,4))=(\bar{p}_1,\bar{p}_2)$

6.1.2. We shall the following notation:

Let S be a set and write $\tilde{P} \subset S$; $\tilde{P}$ designates a collection of subsets of the set S (i.e. $\tilde{P} \subset \exp S$). $\bar{P}$ will designate a partition in S.

Example: $S_1=(1,2,3,4,5)$, $S_2=(1,2,3,4,5,6)$

$\tilde{P}=((1,2,3),(2,4),(5))$ $\tilde{Q}=((1,2,3),(2,4),(5,6))$;

$\tilde{P} \subset S_1$, $\tilde{Q} \subset S_2$, $\tilde{Q} \not\subset S_1$;

$\tilde{A}=((1,2,3);((4,5),(5,6));((2),(3,4)))=(\bar{a}_1;\tilde{a}_2;\bar{a}_3)$

$\tilde{a}_2=(\bar{a}_{21},\bar{a}_{22})$; $\bar{a}_3=(\bar{a}_{31},\bar{a}_{32})$

$\Sigma\tilde{P}$ is the union of all subsets contained in $\tilde{P}$; e.g. for $\tilde{Q}$ given above $\Sigma\tilde{Q}=S_2$.

In the following sections, the topological method, described above is applied to investigations of relationships between ations of adaptive behaviour of Gaines (1972).

6.2. Adaption sets.

In this part we introduce some operations on adaption sets of Gaines[10]. For the purpose of our discussion we have to introduce some new sets concerning non-adaptivity.

6.2.1. The system of closures $\tilde{F}(T_1)$.

Let us define an input-restriction of the automaton $\mathcal{A}$ (1.4.1).

$$\mathcal{A}\upharpoonright(T_1)=\{T_1,P,S,\sigma,\pi\}\ ,\quad T_1\subset T\ .$$

The system of transitive closures (1.3.2) forms an OIMU-topology on the $\mathcal{A}\upharpoonright(T_1)$.

Take all singletons from S and form a collection of closures according to 4.2.2. This partially ordered collection is $\tilde{F}(T_1)$.

6.2.2. Note that for an infinite state-space S we have to use (instead of singletons) all H-connected components of the state-space S.

The problem becomes more complicated. The theory of Čech [12] (pp.362-364,845-846) has to be applied. Also the question, if the next state closure is an A*-topology, has to be decided first.

6.2.3. The most important adaption sets.

$\tilde{F}(t)$	a system of closures of an input restriction by t;
W(t)	the set of satisfactory interaction;
X(t)	the set of unsatisfactory interaction;

The end-sets:

$E(t),\tilde{E}(t)$	the set of the smallest closures (atractors) excluding $\emptyset$;
$E{+}(t),\tilde{E}{+}(t)$	the set of positive attractors (satisf. interact.)
$E'(t),\tilde{E}'(t)$	the set of negative attractors

$O(t), \tilde{O}(t)$	the set of possible adaptivity (it is impossible to access the positive attractors from the set of states which do not belong to this set).
$Q(t), \tilde{Q}(t)$	the set of possible non-adaptivity
$P(t), \tilde{P}(t)$	potentially adaptive
$R(t), \tilde{R}(t)$	potentially non-adaptive
$A(t), \tilde{A}(t)$	adapted (unsatisfactory interaction impossible);
$B(t), \tilde{B}(t)$	non-adaptive (satisfactory interraction impossible;can not adapt)
$C(t), \tilde{C}(t)$	compatibly adaptive
$D(t), \tilde{D}(t)$	compatibly non-adaptive
$J(t), \tilde{J}(t)$	jointly adaptive
$K(t), \tilde{K}(t)$	jointly non-adaptive.

6.2.4. <u>The calculus of adaption sets.</u>

t_i ... single task

T_i ... a set of tasks

$\cap, \cup$ ... set operations on a set of states

$\vee$ ... an operation on inputs

For any $U \in (F,O,Q,P,R,A,B,C,D,J,K)$

$$U(T_i) = \Sigma \tilde{U}(T_i) \; ; \quad \tilde{U}(T_i) = \tilde{F}(T_i) - (\sim U(T_i) \sqsubset \tilde{F}(T_i)) \; ,$$

where dU is a ~~dual set~~ complement of U,

$$\hat{\tilde{E}}_+(T_i) = \tilde{F}(T_i) - \{X(T_i) \sqsubset \tilde{E}(T_i)\} \qquad \tilde{E}'(T_i) = E(T_i) - \{W(T_i) \sqsubset \tilde{\hat{E}}(T_i)\}$$

$J(t) = P(t)$; $K(t) = R(t)$; $J_1(t) = A(t)$ for a single task.

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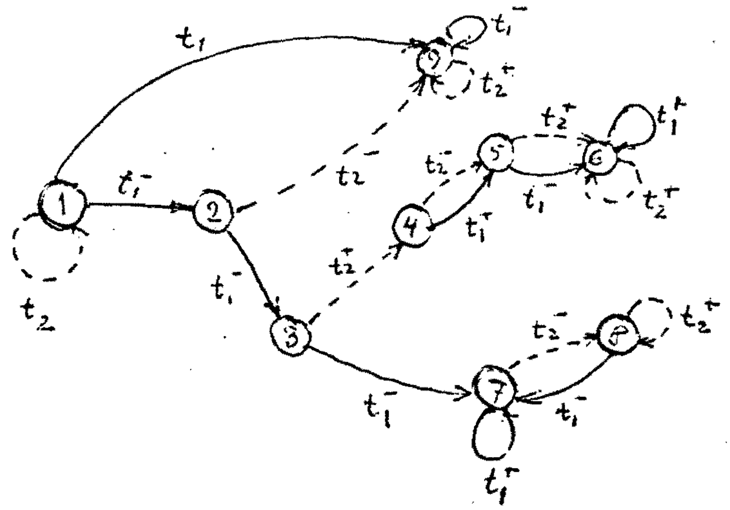
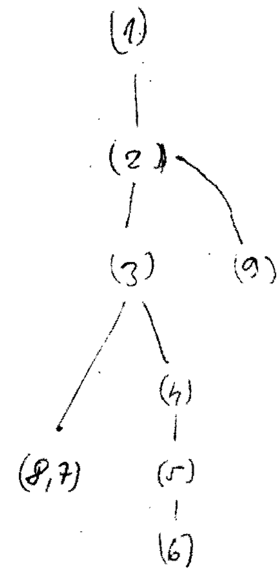
6.3. Examples of application of the calculus.





$$\tilde{F}(t_1) = \left\{ \begin{matrix} (78)\,(7) \\ (1237)(237)\,(37)(7) \\ (456)\,(56)\,(6) \\ (19)\,(9) \end{matrix} \right\}$$

$$\tilde{E}(t_1) = \{(6)\,(7)\,(9)\}$$

$$\tilde{E}^{+}(t_1) = W(t_1) \sqsubset \tilde{E}(t_1) = (467) \sqsubset \{(6)(7)(9)\} = \{(7)(6)\}$$

$$\tilde{E}^{-}(t_1) = \{(9)\}$$

$W(t_1) = (467)$; $X(t_1) = (123589)$

$$\tilde{O}(t_1) = E^{+}(t_1) \sqsubset \tilde{F}(t_1) = \left\{ \begin{matrix} (78)\,(7) \\ (1237)(237)(37)(7) \\ (456)(56)(6) \end{matrix} \right\} ; \quad \tilde{Q}(t_1) = \{(19)(9)\}$$

Let $Inc(t_1) = O(t_1) \cap Q(t_1) = (1)$;

$$\tilde{P}(t_1) = \tilde{O}(t_1) - Inc(t_1) \sqsubset \tilde{O}(t_1) = \left\{ \begin{matrix} (78)\,(7) \\ (237)(37)(7) \\ (456)(56)(6) \end{matrix} \right\} ; \quad P(t_1) = (2345678)$$

$\tilde{A}(t_1) = \{(6)(7)\}$. $A(t_1) = (67)$

$$\hat{F}(t_2) = \begin{Bmatrix} (3456)(456)(56)(6) \\ (78)\ (8) \\ (29)\ (9) \\ (1) \end{Bmatrix} \qquad \check{E}(t_2) = \{(1)(6)(8)(9)\}$$

$$\tilde{E}^+(t_2) = \{(6)(8)(9)\}$$

$$E^-(t_2) = \{(1)\}$$

$$W(t_2) = \{35689\} \qquad X(t_2) = \{1247\}$$

$$\tilde{E}(t_2) = \begin{Bmatrix} (3456)\ (456)(56)(6) \\ (78)\ (8) \\ (29)\ (9) \end{Bmatrix} \; ; \qquad \tilde{P}(t_2) = \check{E}(t_2) \; ; \quad P(t_2) = (23456789)$$

$$\check{A}(t_2) = \begin{Bmatrix} (56)\ (6) \\ (8) \\ (9) \end{Bmatrix} \qquad A(t_2) = \{5689) \qquad \tilde{Q}(t_2) = \{(1)\}$$

$$E'(t_1) \sqsubset \tilde{P}(t_2) = \{(29)(9)\}$$

$$C(t_1 \vee t_2) = P(t_1) \cap P(t_2) - \{(E'(t_1) \sqsubset P(t_2)) \vee (E'(t_2) \sqsubset P(t_1))\} = (345678)$$

$$F(t_1 \vee t_2) = \begin{Bmatrix} (123456789)(23456789)(345678) & (456)(56)\ (6) \\ & (78) \\ (19) & (9) \end{Bmatrix}$$

$$\tilde{C}(t_1 \vee t_2) = \begin{Bmatrix} (345678) & (456)\ (56)(6) \\ & (78) \end{Bmatrix} = \tilde{F}(t_1 \vee t_2) - \left(Q(t_1) \cup Q(t_2) \sqsubset \tilde{F}(t_1 \vee t_2)\right)$$

$$J_A(t_1 \vee t_2) = C(t_1 \vee t_2) - \{\mathrm{compl}(A(t_1) \cap A(t_2)) \sqsubset C(t_1 \vee t_2)\}$$

compl $A = S - A$ i.e. complement.

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$\tilde{O}(T_i)=E+(T_i)\sqsubset\tilde{F}(T_i)$ $\tilde{Q}(T_i)=E'(T_i)\sqsubset\tilde{F}(T_i)$

$\tilde{P}(T_i)=\tilde{F}(T_i)-(R(T_i)\sqsubset\tilde{F}(T_i))$ $\tilde{R}(T_i)=\tilde{F}(T_i)-(P(T_i)\sqsubset\tilde{F}(T_i))$

$\tilde{A}(T_i)=\tilde{I}(T_i)-X(T_i)\sqsubset\tilde{P}(T_i)$ $\tilde{B}(T_i)=\tilde{R}(T_i)-W(T_i)\sqsubset\tilde{R}(T_i)$

$\mathcal{J}(t)=P(t)$; $K(t)=R(t)$; for t a single task.

$C(T_1 \vee T_2)= C(T_1)\cap C(T_2)-(E'(T_1)\sqsubset\tilde{C}(T_2))\cup(E'(T_2)\sqsubset\tilde{C}(T_1))$

$J_A(T)=\tilde{J}(T)-X(T)\sqsubset\tilde{J}(T)$

6.2.5. Generalization of the calculus.

Instead of taking an adaption automaton with a two-element output P=(p+,p') we can take an ordinary automaton with the set of outputs $P=(p_1,p_2,\dots,p_k,\dots)$.

In this case each adaption set will be dependent on the output and the duality between U and dU dissapear. We have to modify the notation accordingly. We shall write e.g. $J(T_i|P_k)$ instead of $J(T_i)$. For our special case of adaption automata we shall obtain $J(T)=J(T|p+)$ and $K(T)=J(T|p')$ etc.

The generalised calculus can be used for investigation of reachibility and controlability in general (non-linear) automata as well as for investigation of some stability properties ...

6.3 Stability

Investigation of relationship between sets describing the state of an adaption automaton and corresponding topologies can be regarded as a special case of investigations of stability. The closures induced by a next-state relation can be compared with the closures induced by the tolerance relation that determins the topology with respect to which the stability of a dynamic trajectory of the automaton is evaluated.

In some instances in may be advantageous to work with the generalised metrics that correspond to the closures in question than with the closures directly.

The tolerance relation that determins the regions of stability can be expressed by a semi-pseudo-metric given by the following expressions:

(1) $d(x,x) = 0$

(2a) $d(x,y) \geqslant 0$

(2b) $d(x,y) = d(y,x)$

The topology induced by an input or a set of inputs defines a preordered standard topological space that can be defined by also by the following generalised measure:

(1) $d(x,x) = 0$

(2a) $d(x,y) \geqslant 0$

(3) $d(x,y) \leqslant d(x,z) + d(z,y)$ pre-ordering

Stability of a set of inputs with respect to to a tolorance topology than can be defined. An automaton is globaly stable with respect to a "tolorance" toplogy if the input-induced preordered topology of this automata is a refinment of the "tolerance" topology. In other words, the automaton is globaly stable if the U-modification of the tolerance topology is finer than the topology induced by inputs.

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In the terms of measures, the necessary condition for the global stability is that the "tolorance" semi-pseudo-metric is also a pseudometric.

A7. The calculus (5.2.4) can be considerably simplified, if we use families of all closed sets as specified in A2 above. (but note that this is disadvantageous in the case we program the calculus on a computer, because (combinatorially) this requires an exhaustive search.

We have to use the operation Σ but otherwise $\cap$ $\cup$ are simple intersections and unions of the elements of a family.

Example: $\mathcal{F}_1 = \{f_1, f_2, f_3, f_4\} = \{(abc)(ab)(c)(bc)\}$
$\mathcal{F}_2 = \{g_1, g_2, g_3\} = \{(ac)(bc)(ab)\}$
$\mathcal{F}_1 \cap \mathcal{F}_2 = \{(ab)(bc)\} = \{f_2, f_4\} = \{g_2, g_3\}$

For the families of closed sets corresponding to the sets S, As, C, N we use the symbols $\mathcal{S}$, $\mathcal{A}s$, $\mathcal{C}$, $\mathcal{N}$.

Similarly, $\mathcal{E}$ will be the family corresponding to the set $\bar{E}(f)$ from 5.2.3.

(exp P denotes the set of all subsets of P)

A8. $\mathcal{E}(T|P)$ is the family of all closed sets of the topology induced by the input & output restriction $A1(T|P)$.

$\mathcal{E}_c(T/P)$ will ~~de~~ denote the family of the end sets corresponding to the set of ~~compo~~ conditionaly stable set.

(In the following example exp(A) denotes the power set of A)

<u>Example:</u> in the case of the adaption automaton from the ~~fig~~ section 5.3.

the $\tilde{\mathcal{E}}_c = \{(6)(78)\}$ and $\mathcal{E}_c = \{(678)(78)(6)\}$.

We can write the following formulae for the families of closed sets of the topolog induced by the i/o restriction $A \uparrow (T|P)$:

$$\mathcal{E}_N(T/P) = (\exp[\sim W(T/P)]) \cap \mathcal{E}(T/P)$$

$$\mathcal{E}_S(T/P) = \exp W(T/P) \cap \mathcal{E}(T/P)$$

$$\mathcal{S}(T/P) = \exp W(T/P) \cap \mathcal{E}(T/P)$$

$$\mathcal{N}(T/P) = \exp(\sim W(T/P) \cap \mathcal{E}(T/P))$$

$$\mathcal{As}(T/P) = \mathcal{E}(T/P) - \mathcal{N}(T/P)$$

For the family $\mathcal{C}(T_1 \vee T_2|P)$ the situation is more complex (this is the mutual conditional stability)

$$\mathcal{C}(T_1 \vee T_2|P) = \text{cover}(\mathcal{E}_c)$$

This can be expressed also by means of H-connectivity (i.e. the reverse of H-separation from 1.2.)

$$\mathcal{C}(T_1 \vee T_2|P) = \text{cover}(\mathcal{E}_{H_{con}} \chi W)$$

where $\mathcal{E}_{H_{con}}$ is the family of sets ~~of~~ mutually connected in $\mathcal{C}(T_1|P)$ and $\mathcal{C}(T_2|P)$. χ denotes the incidence of

A9. Examples.

Adaption automata:
We shall write $\mathcal{A}$ for the family of closed subsets whose cover is A.
Similarly $\mathcal{J}$, $\mathcal{J}_A$, $\mathcal{C}$, $\mathcal{P}$, for families whose covers are J, J_A, C, P.

Then (T is a set of tasks, t_i a single task.)

$$\begin{aligned} \mathcal{S}(T|p^+) &= \mathcal{J}_A(T) \\ \mathcal{S}(t_i|p^+) &= A(T_i) \\ As(T|p^+) &= \mathcal{J}(T) \\ As(t_i|p^+) &= \mathcal{P}(t_i) \\ \mathcal{C}(T_1 \vee T_2|p^+) &= C(T_1 \vee T_2) \end{aligned}$$

the covers will give J_A, A, J, P, C as defined by Gaines.

A10. If we combine tasks, we can compute the new sets directly applying the Theorem 4 from A3. above.

i.e. if $\mathcal{S}(T_1|P) \overset{def}{=} \mathcal{S}_1$, $\mathcal{S}(T_2|P) \overset{def}{=} \mathcal{S}_2$ each defines a different closure

$$\boxed{\mathcal{S}(T_1 \vee T_2|P) = \mathcal{S}_1 \wedge \mathcal{S}_2}$$

Analogically with the other sets.

We work in the lattice of all Čech closures.

For the computational purposes (computer aided syst–) ~~we~~ it is advantageous to use the partial orders from the section 5.2.4.

Control-theoretical interpretation.

5.1.1. Define a free input generator (Gaines q.cit.)

5.1.2. Each free input generator T^* induces a topology on the automaton, which is identical with the topology generated by the input restriction of the automaton A, $A \upharpoonright (T^*)$ under the transitive closure 1.3.2.

5.1.3. The system of closures $\tilde{F}(T)$ of 5.2.1. defines a collection of closed trajectories which are partially ordered by the set inclusion. The set of smallest non-empty elements (the ends of the chain) is the steady-state region of the motion (i.e. sets $\tilde{E}$, E). The rest of the $\tilde{F}$ which cover the smallest (end E) elements represents the transient (i.e. irreversible part of the behaviour under T^*).

5.1.4. If the ~~attest~~ new input gen. T_1^* obtained by enlarging T^*, i.e. $T_1^* = (T^* \vee T_2^*)^*$ does ~~not~~ coincide with the original topology induced by T^* we say that the T_2^* is neutral and the system is insensitive to the disturbance T_2^*. In this case the topology induced by T_2^* is finer ~~than~~ the original topology T^*.

5.1.5. If the topologies ~~are~~ induced by T_2^*, T^* are incompatibles, T_2^* represents a disturbance, which takes the ~~the~~ system out of of the previous steady-state region.

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For other references see Kohout (1975), [9] above.

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THE FUNCTIONAL COMPLETENESS OF PI-ALGEBRAS AND ITS RELEVANCE TO BIOLOGICAL MODELLING AND TO TECHNOLOGICAL APPLICATIONS OF MANY-VALUED LOGICS

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INDEX TERMS

Functional completeness in multi-valued logics, models in biology and medicine, functionally complete Pi-algebras, groupoids, design of functionally complete multi-valued logic systems, infinite valued logics.

ABSTRACT

The paper gives an algebraic formulation of constructive rules for generating the functionally complete sets of functors in many-valued logics. The sets of functors which can be generated by the rules comprise the majority of currently used systems as well as some new systems [5]. The algebraic formulation of the rules, which generalises previous works on Pi-algebras [9],[10], [5], is suitable for treatment of isomorphisms and transformations in many-valued logics.

The paper also examines the role of functional completeness in applications. Particular attention is paid to multi-valued models in biology psychology and medicine as well as to questions of simulation in technology.

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1. INTRODUCTION

Some logicians consider the functional completeness of many-valued logics to be "an interesting feature from the purely formal, algebraic point of view ... in some ways desirable, but certainly not essential" [1]. Indeed, for a logician there exist some other perhaps more important kinds of completeness (e.g. deductive completeness, etc.). It is also true that some important many-valued logics are functionally incomplete.

However, the situation considerably changes if we are interested in applications outside logic. Functional completeness is important in such applications where it is necessary to be able to generate all possible many-valued functions by means of basic connectives. It is obviously desirable to have a set of gates which can generate any combinatory switching circuit [2],[3], but there may exist some other more recent applications of many-valued algebras, where functional completeness is equally if not more important. This question will be briefly dealt with in the next section.

In applications we need to design a complete system of certain properties. This may be difficult if we use the general criteria for completeness of Rosenburg [4]. Some difficulties involved were discussed in [5] together with suggestions as to the use of certain less general rules of a constructive character. These rules, which determine a wide class of complete partially-defined many-valued algebras, were discovered by Pinkava in 1971. These many-valued systems, which are called Pi-algebras in this paper, include most of the currently used many-valued systems, (the Post and lattice systems, ring and semiring systems, the Aizenberg-Rabinovich systems, etc.) as is shown in the table of their partial classification [5] , Fig. 1 . The rules have already been used for generating new sets of gates for many-valued switching networks 5 . More recent applications include the design of a new many-valued functionally complete calculus which is used for analysis of protection structures in multi-user computing systems [6].

The aim of this paper is to give a more general formulation of the rules which because of their algebraic character make it possible to deal with questions of isomorphisms and transformations [7] in a manner similar to that of [8] but for a much wider class of systems.

The results given in this paper generalize the previous results of Pinkava [9], [10] as well as Theorem 10 of Kohout [5].

2. IMPORTANCE OF FUNCTIONAL COMPLETENESS IN SOME APPLICATIONS

Although there exist many types of completeness in logic which may be more important for a logician then functional completeness of logical calculi, functional completeness will often be the primary interest of a scientist or an engineer engaged in practical applications of logic.

The role of functional completeness of logic systems in the design of switching circuits is similar to that of complete spaces in state-space theories of control [11]. A control engineer may wish to use algorithms with guaranteed convergence, and analogically a logic designer would prefer minimalization algorithms converging for any switching function. The latter is impossible if the set of elements into which the switching circuit is decomposed is functionally incomplete.

In pattern recognition using adaptive many-valued logic nets, the set of basic 'cognitive' elements of the net has to be complete, otherwise some possibly important patterns will be misclassified. This is particularly important, if we apply pattern recognition methods based on many-valued logics to a set of medical data 13 for the purpose of medical diagnosis. Similarly, in biological [14] or psychological and medical [15], [16], [17], models based on abstract pattern classification by logics, the choice of an incomplete set of functors as the base of the model would represent a bias towards assumptions which might not be contained in the experimental data. For example, in models of instincts [14], [18], this would represent the a priori assumption that certain forms of instincts do not exist which are already described by the experimental data. In models of psychiatric disturbances [19], [20], this would represent a priori exclusion of some impairments of the structure, diminishing the principal usefulness of the model in the search for new symptoms.

If incomplete systems of logical calculi are used for examining hazards in switching circuits or for modelling fault occurrences in digital circuits, this again leads to the elimination of certain possibly vital hazards and faults from the model, rendering the model unreliable.

In general systems theory approaches to modelling, which use essentially many-valued algebras [3] (an extension of Svoboda's Boolean logic approach [21], [22]), the consideration of functional completeness may influence the choice of sampling mask, which represents a heuristic selection of certain hypotheses.

The common features of all examples hitherto in which functional completeness <u>is important</u> can be summarized in the form of the following principle:

Functional completeness is important in all cases where we are not concerned with deductive systems but with systems which <u>extract</u> some structure from experimental data or where we are concerned with convergence of minimalisation methods applied to many-valued algebraic models.

3. FUNCTIONAL COMPLETENESS IN PI-ALGEBRAS

Definition 4 and Theorems 1,2 represent the main result of the paper. Some auxilliary definitions which are necessary for the proofs and not easily available are given in the Appendix (numbered A1-A7) together with relevant references to the literature.

Definition 1.

Let P be a cyclically ordered set and $a \in P$, $b \in P$.

Let a precede b. We say that b is the direct successor of a iff:

1) $(a,b,c) \in \mathcal{C}$ (for the definition of and other symbols see the appendix).

2) in the set $P \setminus (a)$, b is the first element in $\mathcal{U}_{\mathcal{C}}(a)$

3) in the set $P \setminus (b)$, a is the last element of $\mathcal{U}_{\mathcal{C}}(b)$

Then we write $a \precsim b$. By analogy, we define $b \succsim a$ (b is the direct predecessor of a).

The familiar cyclic negation ($\vec{v}$=v+1 mod k) which was used in previous work can be substantially generalised.

Definition 2. (a cyclic shift function)

Let P be an arbitrary set. If there exists a cyclical order of the set P such that every $p \in P$ has a direct successor q, and q is the direct predecessor of p then we define the cyclic shift function corresponding to that cyclical ordering as a mapping such that;

1) ϕ: $P \to P$

2) for every $p \in P$ it holds that $p \precsim \phi(p)$.

The composition of mappings is defined in the usual way as

$$\phi^{k+1}(p) = \phi(\phi^k(p)) \quad , p \in P$$

Definition 3. (Distance)

Let p_1, $p_2 \in P$ and ϕ be a cyclic shift function. Then the distance δ of the elements p_1, p_2 with respect to ϕ is the least ordinal such that $\phi^{\delta}(p_1) = p_2$. We write $\delta_{\phi}(p_1, p_2)$.

Lemma 1:

Let $\mathscr{C}$ be a cyclical ordering and $\mathscr{C}^*$ its inverse. If p_2 is the direct successor of p_1 in $\mathscr{C}^*$ (i.e. $p_1 \overset{*}{\precsim} p_2$), then $p_2 \precsim p_1$. □

Proof: $J^*(p_1, p_2) = J(p_2, p_1) = 0$ from Def.1 and Lemma A.7 ■

Lemma 2:

is an automorphism. □

Proof: If is not a monomorphism, then for some $p_1 \neq p_k$ $\phi(p_1) = p_j$ and $\phi(p_k) = p_j$, i.e. $p_i \precsim p_j$, $p_k \precsim p_j$. Therefore $J(p_i, p_j) = J(p_k, p_j) = 0$ which is possible only if $p_i = p_k$.

If ϕ is not an epimorphism, then $\phi(x) \neq p_j$ is true for every $x \in P$. Let us take $(p_r, p_j, p_s) \in \mathscr{C}$ and $(p_s, p_j, p_r) \in \mathscr{C}^*$ such that $p_j \precsim p_s$ and $p_j \overset{*}{\precsim} p_r$. From Lemma 1 $p_r \prec p_j$. ■

Definition 4. (Definition of Pi-algebra)

Let Pi be an algebra such that

$Pi = \langle P, \Phi, \odot, \boxplus, \rangle$ where

a) P is its carrier

b) ϕ is a cyclic shift function

c) $\langle G, \Phi \rangle$ is an arbitrary groupoid with zero $z_{(\Phi)}$, without divisors of the zero, and with the absorbing element $a_{(\Phi)}$ such that

$a_{(\Phi)} \Phi p = p \Phi a_{(\Phi)} = a_{(\Phi)}$ for every $p \in P$, $p \neq z_{(\Phi)}$

d) $\langle G, \odot \rangle$ is an arbitrary groupoid with the unit $e_{\odot}$

e) $\langle G, \boxplus \rangle$ is an arbitrary groupoid with a right zero $z_{r\boxplus}$ and a right unit $e_{(r \boxplus)}$.

In order to have a more succinct way of writing let us further introduce the following symbols:

$$\bigoplus_{i=1}^{i=n} \{x_i\} = x_1 \oplus x_2 \oplus x_3 \oplus \dots \oplus x_n \qquad \text{, n finite}$$

Analogically we introduce the symbols $\bigodot_{i=1}^{n}$, $\boxplus_{i=1}^{n}$ for repeated operations $\odot$, $\boxplus$ respectively.

More generally, we shall write e.g. $\bigodot_{x \varepsilon S} x$ for repeated operations of taking all elements from an arbitrary set S.

Let further $\oplus\{\phi^{\varkappa}(v)\} = v \oplus \phi(v) \oplus \phi^2(v) \oplus \dots \dots \oplus \phi^{\xi}(v)$ where ξ is the least ordinal such that $\phi^{\xi}(z_{(\oplus)}) = z_{(\oplus)}$.

$\oplus[\{\phi^{\varkappa}(v)\} - \phi^{\nu}(v)]$ is the abbreviation for
$v \oplus \phi(v) \oplus \dots \oplus \phi^{\nu-1}(v) \oplus \phi^{\nu+1}(v) \oplus \dots$

Definition 5:

$$\psi_{\varkappa}(v) = \begin{cases} a_{(\oplus)} & \text{iff } v = \varkappa \\ z_{(\oplus)} & \text{iff } v \neq \varkappa \end{cases}$$

Lemma 3:

A constant function is given by

$c_{\varkappa} = \phi^{\varkappa}[\oplus\{\phi^{\varkappa}(v)\}]$ where $\varkappa = \delta(z_{(\oplus)}, c_{\varkappa})$, $c_{\varkappa} \varepsilon P$. □

Proof: From Lemma 2 it follows that there will be one and only one function $\phi(v) = z_{(\oplus)}$ for any value of v. Consequently, from the property of the zero, $\oplus\{\phi^{\varkappa}(v)\} = z_{(\oplus)}$. However, $\phi^{\varkappa}(z_{(\oplus)}) = c_{\varkappa}$ ■

Lemma 4:

$$\psi_{\varkappa}(v) = (\oplus)[\{\phi^{\varkappa}(v)\} - \phi^{*\varkappa}(v)]$$

where ϕ^* is an inverse cyclic shift function obtained by the substitution of $\mathscr{C}^*$ (see Def. A6) for $\mathscr{C}$ in Def.2. □

Proof: The expression $\diamondplus[\{\phi^{\varkappa}(v)\} - \phi^{\nu}(v)]$ will assume the value $z_{(\diamondplus)}$ for all argument-values for which one of the functions v, $\phi(v), \phi^2(v)$, $, \ldots, \phi^{\nu-1}(v), \phi^{\nu+1}(v), \ldots$ assumes $z_{(\diamondplus)}$. However, for that argument-value for which $\phi^{\nu}(v) = z_{(\diamondplus)}$ the function $\diamondplus[\{\phi^{\varkappa}(v)\} - \phi^{\nu}(v)] = a_{(\diamondplus)}$, because $\phi^{\nu}(v)$ has been removed and there is always exactly one of the remaining functions to assume the value $a_{(\diamondplus)}$ for that particular value, $\phi^{\nu}(p_j) = z_{(\diamondplus)}$. That $\phi^{\nu}(v) = \phi^{*\varkappa}(v)$ follows from $\phi(z_{(\diamondplus)}) = c_{\varkappa}$.

Lemma 5:

Any function $f(v_1, v_2, \ldots, v_n)$ of a many-valued logic system may be expressed by means of a formula of the following type:

$$f(v_1, v_2, \ldots, v_n) = \bigodot_{\forall (f(a_1, a_2, \ldots a_n) | f \neq e_{(\odot)})} \phi^{\delta_2}(\phi^{\delta_2^*}(c) \boxplus \phi^{\delta_1}[\diamondplus_{\ell=1}^{\ell=n} \psi_{a_\ell}(v_\ell)])$$

where $\delta_1 = (a_{(\diamondplus)}, e_{(\boxplus)})$, $\delta_2 = (z_{(\boxplus)}, e_{(\odot)})$, and $\delta(z_{(\diamondplus)}, z_{(\boxplus)}) = \delta_1$. □

Remark: In the above formula $\bigodot_{\forall (f \ldots)}$ means the repeated operation for such substitutions $a_1, a_2, \ldots, a_n$, $a_j \in P$, of the variables $v_1, v_2, \ldots, v_n$, for which $f \neq e$.

The distance δ_2^* is the inverse of the distance δ_2.

Proof: Any many-valued function depending at least formally on n variables may be viewed as a family of all (n+1)- tuples of the type $\langle a_1, a_2, \ldots, a_n; \underline{b} \rangle_\lambda$ where $b \in P$. Here, $\langle a_1, a_2, \ldots, a_n \rangle_\lambda$ are the individual substitutions of the variables $v_1, v_2, \ldots, v_n$ causing $f(v_1, v_2, \ldots, v_n) = b_\lambda$. For the substitution $\langle a_1, a_2, \ldots a_n \rangle_\lambda$ the respective set of characteristic functions $\langle \psi_{a_1}(v_1), \psi_{a_2}(v_2), \ldots, \psi_{a_n}(v_n) \rangle_\lambda$ will assume the value $a_{(\diamondplus)}$, and $z_{(\diamondplus)}$ for other substitutions. Therefore $\langle \psi_{a_1}(v_1) \diamondplus \psi_{a_2}(v_2) \diamondplus \ldots \diamondplus \psi_{a_n}(v_n) \rangle_\lambda = a_{(\diamondplus)}$ when substituted

$\langle a_1, a_2, \ldots, a_n \rangle_\lambda$ and $z_{(\Phi)}$ otherwise.

Further, $\phi^{\delta_1}[\Phi\{\psi_{a_\ell}(v_\ell)\}] = e_{(r\boxplus)}$ for that substitution and $\phi^{\delta_1}(z_{(\Phi)}) = z_{(\boxplus)}$ otherwise.

If then we form an expression of the type

$c_\gamma \boxplus \phi^{\delta_1}[\Phi\, \psi_{a_\ell}(v_\ell)_\lambda$ then owing to the existence of $e_{(r\boxplus)}$, the expression will assume the value b for that substitution, and $z_{(\boxplus)}$ otherwise if we set $c_\gamma = b_\lambda$. If we add the transformation of the constant, we obtain instead of b_λ the value $\phi^{\delta_2^*}(b_\lambda)$. This value will be transformed back to b_λ after the application of the last transformation

$\phi^{\delta_2}(z_{(\boxplus)}) = e_{(\odot)}$

From the properties of $\odot$ it follows, that the whole formula will assume the same values $b_\lambda \neq z_{(\Phi)}$ as the function $f(v_1, v_2, \ldots, v_n)$ for each respective substitution $\langle a_1, a_2, \ldots, a_n \rangle_\lambda$. ■

<u>Theorem 1:</u>

Any Pi-algebra is functionally complete if the following condition is satisfied:

$$(z_{(\Phi)}, z_{(\boxplus)}) = \delta_1, \text{ where } \delta_1 = (a_{(\Phi)}, e_{(\boxplus)}).$$

<u>Proof:</u> It easily follows from Lemmata 3,4,5.

<u>Theorem 2:</u>

If the right zero $z_{(r\boxplus)}$ is also the zero and the right unit $e_{(r\boxplus)}$ is also the unit of the groupoid $\langle G_j, \boxplus \rangle$, then the following holds:

$$\underset{\forall (f \neq e_{(\odot)})}{\odot} \phi^{\delta_2}(\phi^{\delta_2^*}(c_\gamma) \boxplus \phi^{\delta_1}[\underset{\ell=1}{\overset{\ell=n}{\Phi}} \psi_{a_\ell}(v_\ell)]) = \underset{\forall (f \neq e_{(\odot)})}{\odot} \phi^{\delta_2}(\phi^{\delta_2^*}(c_\gamma) \boxplus [\underset{\ell=1}{\overset{\ell=n}{\boxplus}} \phi^{\delta_1}(\psi_{a_\ell}(v_\ell))])$$

Proof: The functions $\phi_{a_\ell}(v_\ell)$ can assume only values $z_{(\Phi)}$, $a_{(\Phi)}$. Thus in this case ⊞ will behave after the transformation ϕ^{δ_1} in the same way as Φ before the transformation.

It is obvious that the majority of the theorems from [9], [10], as well as the classification of the Pi-algebras given in [5] remain valid for the generalisations given in this paper.

4. CONCLUSION

The aim of this paper is to provide some sufficient conditions for cunctional completeness of a constructive character, which can be used for the design of complete sets of functors to suit the requirements of various applications.

It has been noted that the general conditions of Rosenberg are particularly useful in establishing negative results [23] but in order to obtain positive results in an easy way some more constructive methods have to be developed.

It should be noted that the conditions given in this paper can also be used for establishing positive results concerning functional completeness in algebraic systems. The groupoids of the complete Pi-algebra can be 'collapsed' in various ways into a single functor which makes it possible to obtain results similar to that of [24].

APPENDIX

We shall give here some definitions and theorems used in this paper. The proofs can be found in the literature quoted.

Definition A1: (Čech [25] p.34)

A cyclical ordering of a set P is a subset $\mathcal{C}$ of the set PxPxP satisfying the following conditions:

1) $(a,b,c)\in\mathcal{C} \Rightarrow (b,c,a)\in\mathcal{C}$
2) $(a,b,c)\in\mathcal{C}$ and $(b,a,c)\in\mathcal{C}$ never hold simultaneously
3) if neither $(a,b,c)\in\mathcal{C}$ nor $(b,a,c)\in\mathcal{C}$, then two of the elements a,b,c are equal
4) $(a,b,c)\in\mathcal{C}$, $(a,c,d)\in\mathcal{C} \Rightarrow (a,b,d)\in\mathcal{C}$

Lemma A2:

Let P be a cyclically ordered set and let a P. If $x\in P\setminus(a)$, $y\in P\setminus(a)$, we say that x precedes y if and only if $(a,x,y)\in\mathcal{C}$. Then we shall write $x\prec y$. This defines an ordering of the set $P\setminus(a)$, which will be denoted $\mathcal{U}_{\mathcal{C}}(a)$. For the proof that this is an ordering see Čech [25] p.34.

Lemma A3:

Let $a\in P$ and let there be given an ordering of the set $P\setminus(a)$. Then there is exactly one cyclical ordering $\mathcal{C}$ of the set P such that the given ordering of the set $P\setminus(a)$ coincides with $\mathcal{U}_{\mathcal{C}}(a)$.

Proof: see Čech [25], p.35

Definition A4:

Let P be a cyclically ordered set and let a∈P, b∈P, a≠P. Denote by $J_{\mathcal{C}}(a,b)$ the set of all x such that $(a,x,b) \in \mathcal{C}$. This set is called an interval of the set P, with beginning a and end b.

Definition A5:

Let $G_{(*)} = \langle P, * \rangle$ be a groupoid with the domain P and the groupoid operation *. Then the left zero element $z_{l(*)}$ is defined by $z_{l(*)} * p = z_{l(*)}$; a left unit element $e_{l(*)}$ is defined by $e_{l(*)} * p = p$; iff the equalities hold for every p∈P. A left absorbing element $a_{l(*)}$ is defined by $a_{l(*)} * p = a_{l(*)}$ if it holds for some p∈P. By analogy, the right zero $z_{r(*)}$ is defined by $z_{r(*)} = p * z_{r(*)}$. Similarly we can define a right unit or a right absorbing element. The zero $z_{(*)}$ (or a unit, an absorbing element) is both, the left zero (left unit, a left absorbing element) and the right zero (a right unit, a right absorbing element).

Definition A6:

Let $\mathcal{C}$ be a cyclical ordering of a set P. Define $\mathcal{C}^* \subset P \times P \times P$ as follows:

$(a,b,c) \in \mathcal{C}^* \Leftrightarrow (c,b,a) \in \mathcal{C}$

Then $\mathcal{C}^*$ is the inverse cyclical ordering to $\mathcal{C}$.

Lemma A7:

Let $\mathcal{C}, \mathcal{C}^*$ be two mutually inverse cyclical orderings of P. Then $J_{\mathcal{C}^*}(a,b) = J_{\mathcal{C}}(b,a)$, where a,b∈P, a≠b.

A more detailed exposition of cyclically ordered spaces (including infinite topological spaces) is given in [26] , pp. 294-312.

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Application of Fuzzy Logic to Controller Design Based on Linguistic Protocol

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Abstract

This paper describes an application of fuzzy logic in designing controllers for industrial plants. In such cases where a linguistic control protocol is easily derivable or exists in the mind of a skilled operator a Fuzzy Logic can be used to synthesize this protocol. Fuzzy logic transforms the imprecise linguistic statements into a precise numerical calculus which can be used by an on-line computer. The method has been applied with success to pilot scale plants as well as in a practical industrial situation. The merits of this method in its usefulness to control engineering are discussed.

Fuzzy logic is able to tackle sets of linguistic statements which describe the interrelation between several variables.

This ability could possibly be exploited in other fields apart from control engineering, which have not been investigated sufficiently because of their unsuitability to treatment by a precise branch of mathematics. The work, therefore, illustrates the potential for using fuzzy logic in modelling and in "soft" applications like decision-making.

The success of implementing a fuzzy controller depends, of course, on the availability and reliability of the linguistic protocol. This cannot always be guaranteed except in simple cases because the protocol may be too complicated for the operator to reproduce in whole and accurately or it is too difficult to develop because of the complexity of the process. The objective or goal that is to be achieved, however, is often much easier to verbalize and consequently the need to go beyond a purely descriptive approach and explore means by which a prescriptive system may be implemented presents itself. Possible implementations of such a method are described.

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1. Introduction

The fact that mathematics as a whole is taken to be synonymous with precision has caused many scientists and philosophers to show considerable concern about its lack of application to real world problems. This concern arises because in logic as well as in science there is constantly a gap between theory and the interpretation of results from the inexact real world. Many eminent thinkers have contributed to the discussion on vagueness, occasionally holding human subjectivity as the culprit.

In an excellent analysis of the subject Black [1] says... "that with the provision of an adequate symbolism the need is removed for regarding vagueness as a defect of language". In his paper he strongly argues that vagueness should not be equated with subjectivity. Briefly, his argument may be summarised by noting that the colour 'Blue', say, is vague but not subjective since its sensation among all human beings is roughly similar. It is possible to deal with colour precisely by considering the e.m. radiation producing it but in doing so the important human sensation of colour, as it happens to be vague, has to be sacrificed. Furthermore, it may be argued that vagueness is not a defect of language but also important source of creativity. Analogies are extremely important to creative thinking and vagueness plays a dominant role in such thought process.

Black's motivation to symbolise vagueness appears to be at the back of all investigations of "Deviant Logics" [2]. An important contribution in the past 10 years has been that of

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Zadeh's fuzzy-set-theory and fuzzy-logic [3]. In his recent writings Zadeh [4,5] states clearly his motivation which is to use fuzzy sets to symbolise Approximate Reasoning (AR). Whereas there are many applications of fuzzy-set-theory, this paper describes one of the first results in the application of AR and linguistic synthesis.

1.1. *An Outline of the Paper's Content:* The intention in this paper is to review the whole program of investigation concerning the application of Fuzzy-logic to controller design and to analyse the findings in order to offer insightful comments and conclusions. The original work in this program was done in early 1974 [6] and first published later that year [7,8]. This was the control of a pilot scale steam-engine using fuzzy-logic to interpret linguistic rules which qualitatively express the control strategy. This work is briefly reviewed in the next section of this paper.

Since the publication of the above work several researchers, elsewhere have also implemented the approach using different pilot scale plants. This together with the continuing work as part of this programme have produced results which throw more light on the usefulness of applying fuzzy-logic to linguistic synthesis. Section 3 below offers comments on some of the key findings of these studies.

One of the comments that has been made about fuzzy-logic is that in its present form it is essentially descriptive and does not offer a prescriptive approach to reasoning. In the first place, it should be noted that fuzzy-logic, like any other form of logic, can only be a system for inferring consequences from previously stated premises and only from these premises. A prescriptive system is possible, however, if a hierarchical decision making approach is used so that the strategy at a lower level is derived as a consequence of a description at a higher level. Two early implementations of such a prescriptive method (some might term this a learning or an adaptive approach) are discussed in section 4 of this paper. To conclude this paper the last section examines the future trend in this field in the light of experience being gained from current investigations described here.

2. An Experiment in Linguistic Synthesis

2.1. A Brief Review of fuzzy-logic; The point of view adopted here is that the variables are equated to universes of discourse which are non-fuzzy sets. These variables take on specific linguistic values. These linguistic values are expressed as fuzzy subsets of the universes.

Given a subset A of X ($A \subset X$) A can be represented by a characteristic function: $X_A: X \to \{0,1\}$. If the above mapping is from X to a closed interval [0,1] then we have a fuzzy subset. Thus if A were a fuzzy subset of X it could be represented by a membership function: $\mu_A: X \to [0,1]$.

Note that X is a non-fuzzy support set of a universe of discourse, say, height of people. A can then be equated to a linguistic value such as tall people. Fig. 1 shows two linguistic values A_1 and A_2 and their logical combinations $\bar{A}_1$; $A_1 \wedge A_2$; $A_1 \vee A_2$; where:

$\bar{A}_2$ is formed by taking $(1-\mu_{A_2})$ as membership value at each element of support set,

$A_1 \wedge A_2$ is formed by taking min (μ_{A_1}, μ_{A_2}) at each element of support set, and

$A_1 \vee A_2$ is formed by taking max (μ_{A_1}, μ_{A_2}) at each element of support set

It is in the definition of implication that this logic may be found to differ from other logics. Given $A \rightarrow B$ (If A then B), then it can happen that A and B are linguistic values of two disparate universes of support say X and Y. Note that here the implication is between individual values and not the underlying variables. Thus the relation R between A and B is a fuzzy subset of the universe of support X x Y, the cross-product of X and Y. $\mu_R: X \times Y \rightarrow [0,1]$. $\mu_R(x \times y)$ is related to $\mu_A(x)$ and $\mu_B(y)$ (in the present application) by the following:

$$\mu_R(x \times y) = \min (\mu_A(x), \mu_B(y)).$$

If the relation R represents a "nested" implication (i.e. If A then (If B then C) or $A \rightarrow B \rightarrow C$), then R will have a correspondin higher order cross-product support set.

Now if some relation R between A and B is known and so is some value A^1 then the idea is to infer B^1 from R and A'; $B^1 = A^1$ oR, where A^1 is composed with R. This has the effect of reducing the dimensionality of the support set of R to that of B^1. In this work, the compositional rule of inference used to relate μ_{B^1} to μ_R and μ_{A^1} is:

$$\mu_{B^1}(y) = \max_x \min (\mu_{A^1}(x), \mu_R(x \times y)).$$

These definitions are themselves a matter of much discussion but that concern is outside the scope of this paper. The setting up of relations R from stated implications between fuzzy values and the subsequent use of the rule of inference are the chief mechanism used in decision making in the application described below.

2.2 Application to fuzzy-controllers: As stated earlier the linguistic synthesis approach was initially applied to control a pilot scale steam-engine, a more detailed description of which is given elsewhere [6,7,8]. A concise summary of this work is presented here. The overall control system is shown in figure 2. One aspect of control in this system is the regulation of pressure in the boiler around a prescribed set-point. The control is achieved by measuring the pressure at regular intervals and inferring from this the heat setting to be used during that interval. The essence of this work is simply that if an experienced operator can provide the protocol for achieving such a control in qualitative linguistic terms, then fuzzy-logic as described above can be used to implement successfully this strategy.

The protocol obtained from the operator in this case considers *pressure error* (PE) and *change in the pressure error* (CPE) to infer the amount of *change in the heat* (HC). The protocol consists of a set of rules in terms of specific linguistic values of these variables and is shown in figure 3*. Now it can be seen that these rules are in the form of If...Then statements (implications) and thus, from above, each rule i will translate into a relation R_i. The overall protocol is then a relation R formed by 'oring' together the R_i's:

$$R=R_1 V R_2 \ldots V R_i \ldots V Rn.$$

Let us say now that each rule R_i represents an implication $A_i \rightarrow B_i \rightarrow C_i$. The decision making algorithm that is implemented

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*The abbreviations used for these linguistic values here are: ZE-zero; PZ-positive zero; PS-positive small, PM-positive medium; PB-positive big and the same for negative values NZ,NS,NM and NB. Change in Error negative is taken as movement towards set-point and positive as away from set-point.

contains two phases:

a. The initial setting up phase when the protocol R is formed from two sets of data:

(i) The individual linguistic values A_i, B_i, C_i given as fuzzy subsets.

(ii) The rules as in fig. 3 which specify the actual combination of these values to form each R_i.

b. The decision making phase is invoked at each sampling instant during run-time with the exact measured values A^1 and B^1 supplied to it. This phase then is nothing but the use of compositional rule of inference to derive C^1 as follows:

$$C^1 = A^1 o (B^1 o R).$$

Note that A^1, B^1 can be non-fuzzy, whereas since C^1 is a fuzzy subset of the set of all possible actions, a procedure is required to determine the actual action to be taken from the knowledge of C^1. Also there is a certain advantage in deferring the computation of R until the second phase. Because then this provides a means of altering the control strategy on-line by altering the data structures containing the rules during run-time. However, what need concern us at present is the results obtained from the application of this method to the pilot scale plan. In repeated trials it was found that the results compared favourably with those from applying classical methods from control engineering practice (i.e. 2 or 3 term controllers).

3. Comments on Fuzzy-logic Controller Studies

Two main conclusions have been drawn from this work.

First, that the results vindicate the approach advocated by Zadeh and demonstrate its potential. Second, it can be asserted that the method can easily be applied to many practical situations. This assertion is supported by considering a practical instance, that of cement kiln operation, in which a similar control protocol obtains. In a book on cement kilns, Peray and Waddell [9] list a collection of rules for controlling a kiln. Examples of these rules are shown in figure 4. From this it is immediately apparent that the method as described, can be used for translating these rules. Furthermore, this method has also been tested on plants such as batch chemical reactors, heat-exchangers and so on. Some key feature emerging from these studies are mentioned here for the sake of interest.

In many of the studies, rules exactly as those given in fig. 3 are used with only minor changes. This is not surprising as the rules indicate the relationship between *error*, *change in error* and *control action* that exists in most dynamical plants. This relation is mainly one of monotonicity between the outputs of a plant and the input applied to it. What is more of interest is that in most studies it is found that this form of controller is far less sensative to parameter changes within the plant than the classical 2-term controller. At this stage only a qualitative explanation can be offered for this. It appears that the former is a *reasonable* controller as it relies on the underlying relationships between the plant outputs and inputs whereas the latter is a *pedantic* controller in which the action is computed as a linear combination of the measurements and thus more susceptible to parameter changes.

It is the first conclusion above, however, which is more important. Approximate Reasoning approach outlined here is obviously applicable to other areas as well. The one that has been considered is the design of traffic signal controllers. Application to more obvious areas of decision making in complex and humanistic system will no doubt be attempted in future. If the method described above is applied to these other areas then the likely sources of difficulties to be encountered can be attributed to one main factor. This is that the quality of decision is only as good as the relation R from which it is inferred. R in turn is affected by three factors.

First, it is affected by the set of rules in the protocol. With more complex situations a good protocol is not easy to derive. A great deal of investigatory effort normally referred to as human factors in control is devoted to exactly such matters. Unlikely as it may seem, the human being does not always find it easy to verbalise his considerations during decision making. The only mitigating factor here is that it is far more difficult to determine the decision heuristics in a form amenable to treatment by a branch of precise mathematics than it is to derive rules for linguistic synthesis.

Second factor affecting the quality of decision (though not R itself explicitly) is the underlying range of elements in the support set which provides the context for interpreting the linguistic rules. This can be illustrated by noting that 'tall people' in a land of pygmies is likely to have the support set of range of height from 3 to 5 ft. 6in say whereas the more normal range of height may be say from 4ft. to 7ft. Such considerations are implicit in any application and are equivalent to what a control engineer would term the gains applied to each variable.

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Finally, R is affected by the membership values in the fuzzy subsets defining the linguistic values. This is perhaps the least important of all the factors because the degree of change permitted here is limited as too much change in the membership values of a fuzzy subset is likely to affect the linguistic meaning ascribed to it. This is illustrated in figure 5 in which the effect of given linguistic value (bold line) is altered by using a different linguistic value (as in a), increasing the gain thus decreasing the range of the support set (as in b) and lastly in a minor way by adjusting the defining values of the fuzzy subset.

4. A Recipe for a Prescriptive Approach

4.1. An Early Implementation

As mentioned earlier, the main difficulty that arises is that a good decision requires that a good set of rules are described at the beginning. In any application of reasonable complexity this is not easy to achieve. Indeed it is quite possible that for some reason a protocol cannot be obtained at all. This may be due to the complexity of the plant (e.g. non-linearities) or to the fact that the operator cannot verbalize his decision process adequately or no consistent protocol can be found. However, the goal in any application and a set of assumptions regarding that application can often be much easier to state. This fact motivates investigations into so called learning or adaptive systems.

In the control situation the goal is simply to bring the output to the set-point and keep it there, the only assumption being that the plant input and output are monotonically related. This monotonicity relation enables wrong control actions to be corrected. If the output is too high then too much input was applied and vice versa and so the proper amount of input required can usually be inferred backwards from the stated goal. An early attempt at implementing one such prescriptive system [redacted] is described here.

The overall schematic diagram for the control system is shown in figure 6. The whole system consists of two heirarchical levels. The higher contains the goal which is effectively a bound within which the output is to be maintained. This band, figure 7, is specified by a set of fuzzy rules whose input is time from the start of control and set point deviation i.e. the error signal. The output from the rules specify the changes to be made in the controller. For example:-

a) IF Time is Small AND Error is Negative Big THEN Desired change is Big.

b) IF Time is Big AND Error is Positive Zero THEN Desired change is Zero.

The band can therefore be viewed as a set of local performance criteria which the response must satisfy.

The output from these "teaching" rules alters the lower level control rules appropriately. Since the control rules are of the form $A_i \rightarrow B_i \rightarrow C_i$, the modification is effected by first finding the linguistic values A_i and B_i which best describe the plant state for which a change in action is desired. This search is simply carried out by a supremum operation over the range of linguistic values. The action, C_i, corresponding to that control rule is altered by the amount given by the "teaching" algorithm. If no such rule exists then one is generated.

Figure 8. shows results obtained from applying this scheme. The tables are a method of displaying all the linguistic rules

of the controller. The measurement error and change in error are given on the axes and the entries indicate the actions applied. The abbreviations are as stated in the footnote on page 5.

The rules in figure 2(a) (without the asterisk) are the best designed rules of an experienced operator. On applying the above procedure the controller converges (i.e. there are no more requests for modification of rules) by creating the extra rules marked with an asterisk. Convergence in this sense means that if the system under question is controllable within the prescribed band, then the rules will converge to a solution in a finite number of training steps. On starting with no rules at all and then applying this procedure, convergence takes place to the set of rules shown in figure 2(b). The output trajectory was observed to be marginally better in the second case. The output response of both these policies fit the prescribed band. When the band is narrowed then no convergent policy is found but the response tends to remain within the band. This lack of convergence could be attributed to the 'credit assignment' problem which could be tackled by the 'bootstrapping' technique. Furthermore, lack of convergence could also be attributed to the failure in including sufficient state variables of the plant in the controller.

These modifications are currently being included and are the subject of further experiments.

4.2. An Alternative Approach

The prescriptive approach described above is very much an ad hoc implementation. It serves to illustrate what needs to be done to go beyond a simply descriptive system. What is desired is that such an approach should appear naturally in a suitably improved fuzzy logic theory itself. This is especially relevant to the way in which a change demanded by the higher or 'teaching' algorithm is transmitted to the lower one.

The general philosophy of this approach is shown in figure 9. which depicts a very general learning system. The concept of the membership function enables the set of rules to be expressed as a data base and the linguistic input value as input data. All operations that are required are carried out on this numerical data and are retranslated into a linguistic output value or a new set of rules only when it is necessary to present the results. The idea of the membership function interfaces between the imprecise heuristics and the exact numerical data which describes them.

In the early implementation described above the method of transmitting the change from the higher to the lower algorithm was achieved by reverting to the linguistic rule text and substituting certain linguistic names by others or generating new ones. A much more direct approach [redacted] is to perform numerical operations on the overall relation matrix which describes the controller according to the change demanded by the teaching algorithm.

The controller relation matrix, R_{ABC}, is changed as follows. If for an input A^1 and B^1 an action $\hat{C}$ is required instead of C^1 then the relation $A^1 x B^1 x C^1$ is removed from R_{ABC} and a new relation $A^1 x B^1 x \hat{C}$ is included. To effect this the theory of fuzzy logic is extended to include operations between relation matrices as well as fuzzy sets.

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Once the controller has converged and no more learning takes place it is necessary to develop the new rules as figure 9. indicates. The new relation together with the old spreads and the new ones generated are input to a program which outputs the sets belonging to each rule having first performed a minimization.

This new approach appears more general and direct than the earlier one and experimental results (with a batch reactor plant) have so far proved encouraging.

5. Conclusions

The two prescriptive approaches described above are the first step in an attempt to advance further than a purely descriptive system using fuzzy-logic theory. If, as is suggested here, hierarchical statements are a main requirement of such a theory then this means that fuzzy-logic should have an auto-descriptive property found in multiple valued logics [10]. From the application point of view both a learning situation described here as well as decision making in complex systems are best framed in terms of hierarchical structures. This is very much the direction in which the theory of fuzzy-logic and approximate reasoning is likely to go. The work described in this paper demonstrates the great potential of applying fuzzy-logic theory not only to control engineering problems but also to management and other humanistic systems.

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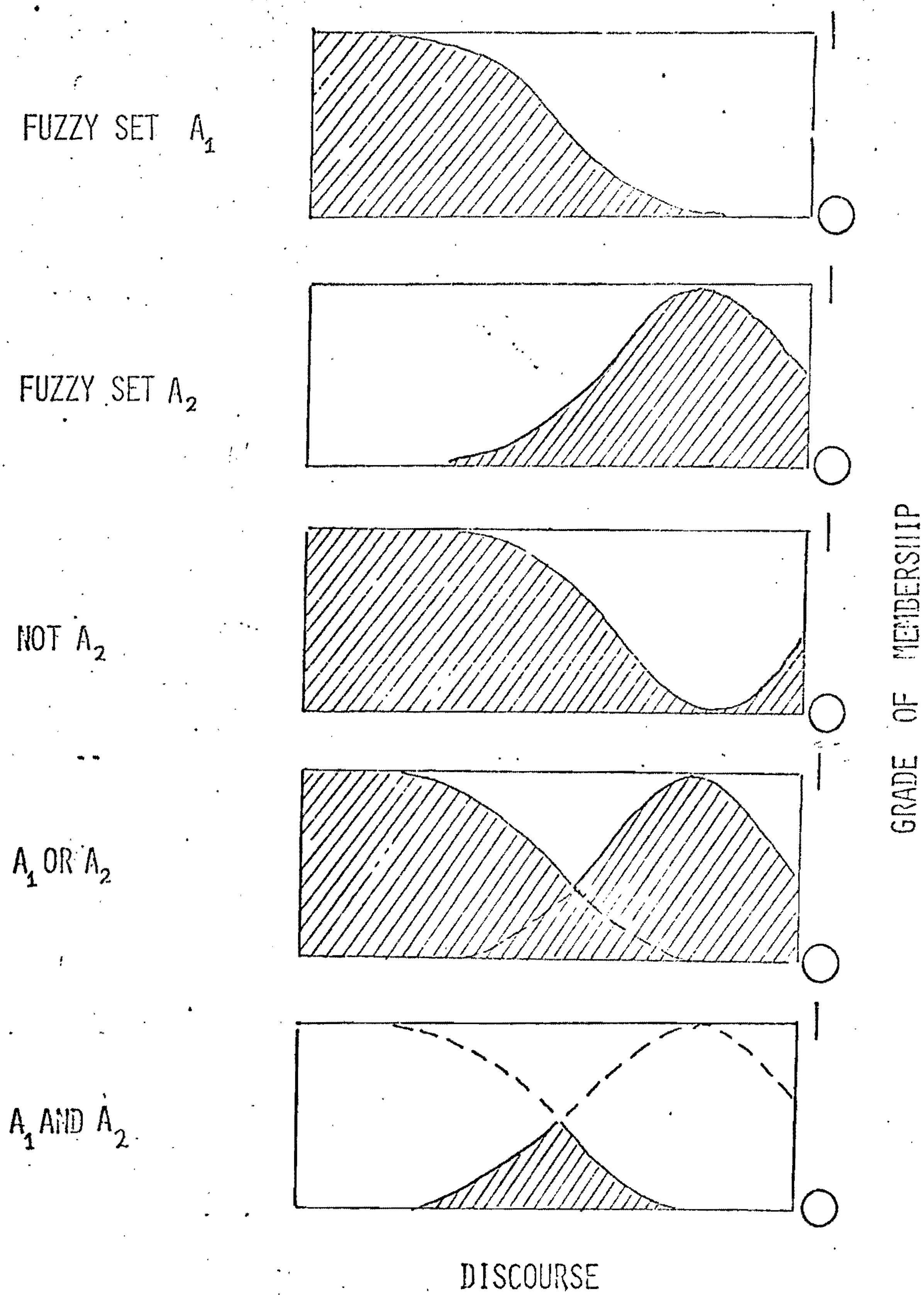
FUZZY SET A1
FUZZY SET A2
NOT A2
A1 OR A2
A1 AND A2
1
0
GRADE OF MEMBERSHIP
DISCOURSE

FIG.1

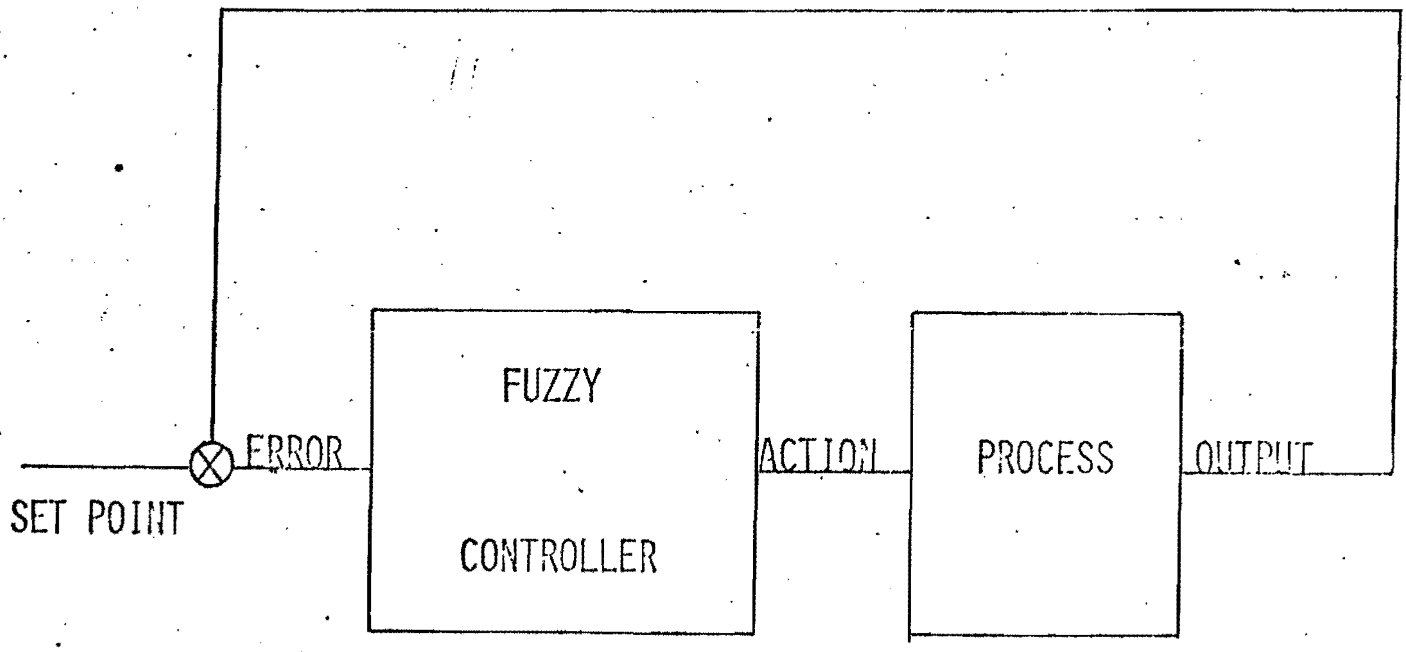

FIG.2 FUZZY LOGIC CONTROL SYSTEM

PRESSURE ERROR = PE , CHANGE IN PRESSURE ERROR = CPE
AND HEAT INPUT CHANGE = HC

IF PE = (NB OR NM) THEN IF CPE = NS THEN HC = PM

OR

IF PE = NS THEN IF CPE = PS THEN HC = PM

OR

IF PE = NO THEN IF CPE = (PB OR PM) THEN HC = PM

OR

IF PE = NO THEN IF CPE = (NB OR NM) THEN HC = NM

OR

IF PE = PO OR NO THEN IF CPE = NO THEN HC = NO

OR

IF PE = PO THEN IF CPE = (NB OR NM) THEN HC = PM

OR

IF PE = PO THEN IF CPE = (PB OR PM) THEN HC = NM

OR

IF PE = PS THEN IF CPE = (PS OR NO) THEN HC = NM

OR

IF PE = (PB OR PM) THEN IF CPE = NS THEN HC = NM

FIG.3

BACK-END TEMPERATURE = BE , BURNING ZONE TEMPERATURE = BZ
PERCENTAGE OF OXYGEN GAS IN THE KILN EXIT GAS = OX

CASE	CONDITION	ACTION TO BE TAKEN
1	BZ LOW	WHEN BZ IS DRASTICALLY LOW
	OX LOW	A. REDUCE KILN SPEED
	BE LOW	B. REDUCE FUEL
		WHEN BZ IS SLIGHTLY LOW
		C. INCREASE I.D. FAN SPEED
		D. INCREASE FUEL RATE
2	BZ LOW	A. REDUCE KILN SPEED
	OX LOW	B. REDUCE FUEL RATE
	BE OK	C. REDUCE I.D. FAN SPEED
3	BZ LOW	A. REDUCE KILN SPEED
	OX LOW	B. REDUCE FUEL RATE
	BE HIGH	C. REDUCE I.D. FAN SPEED

TOTAL OF 27 RULES

FIG.4

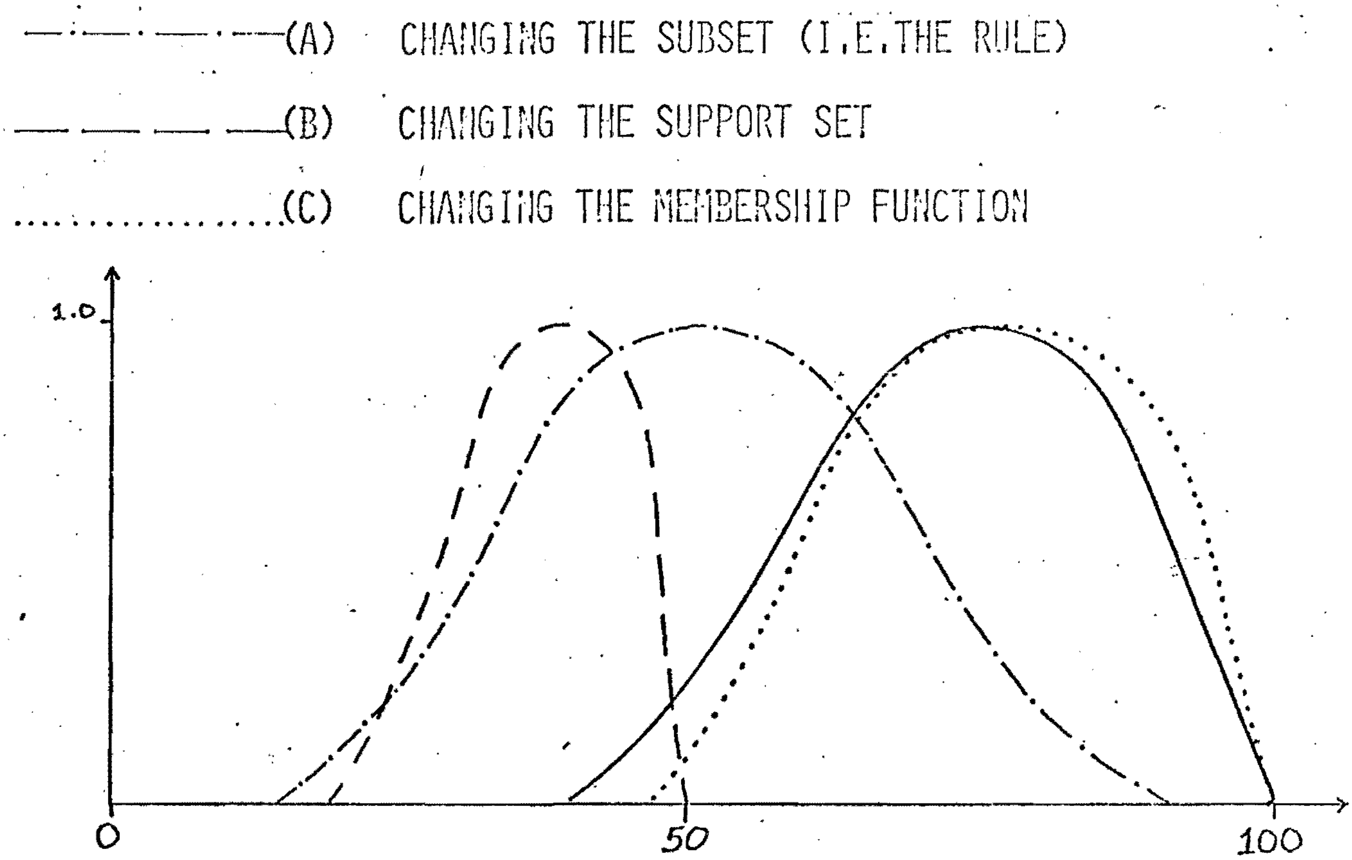

FIG.5 MODIFICATION OF FUZZY CONTROL

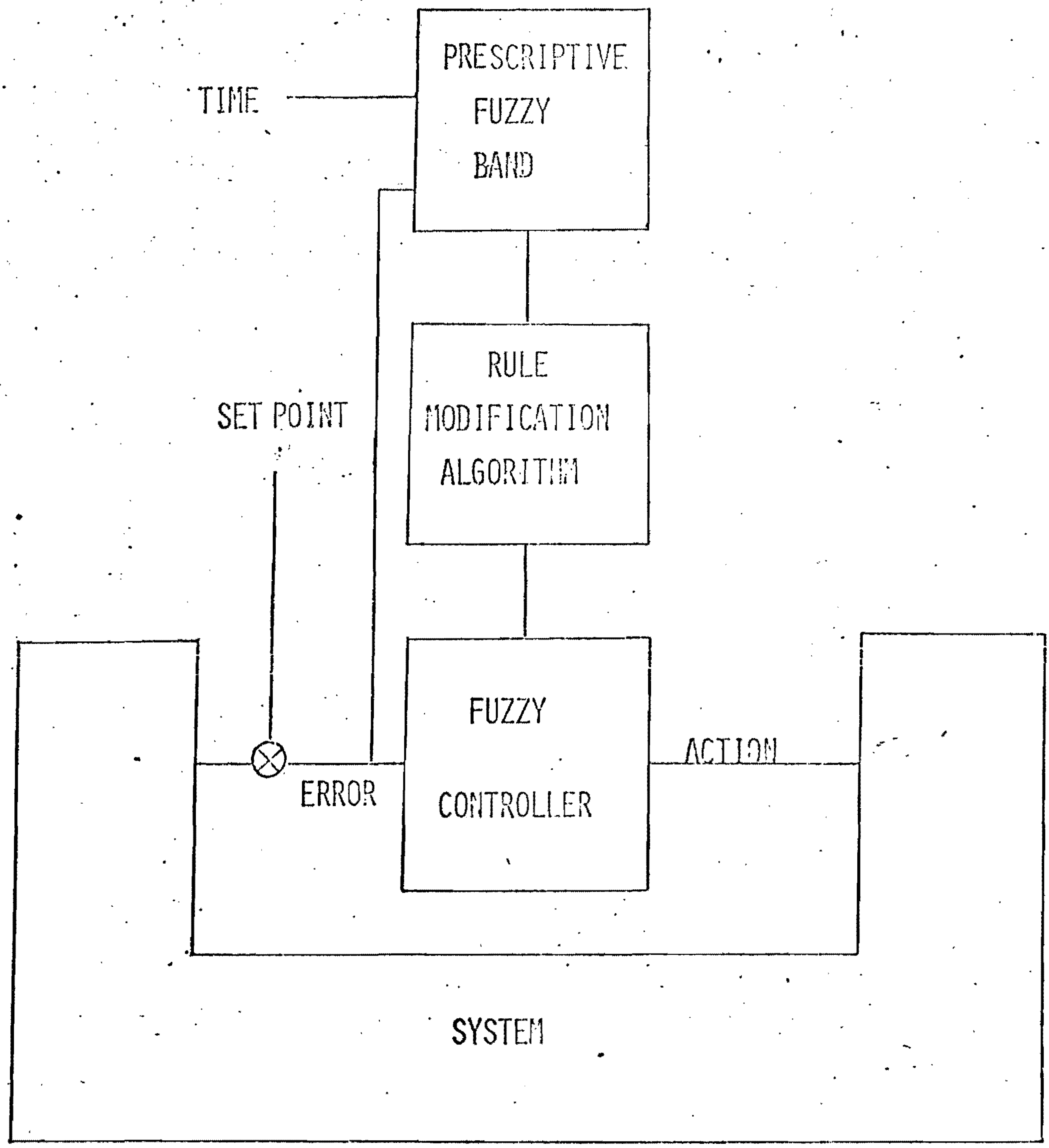
PRESCRIPTIVE FUZZY BAND
TIME
RULE MODIFICATION ALGORITHM
SET POINT
FUZZY CONTROLLER
ERROR
ACTION
SYSTEM

FIG.6

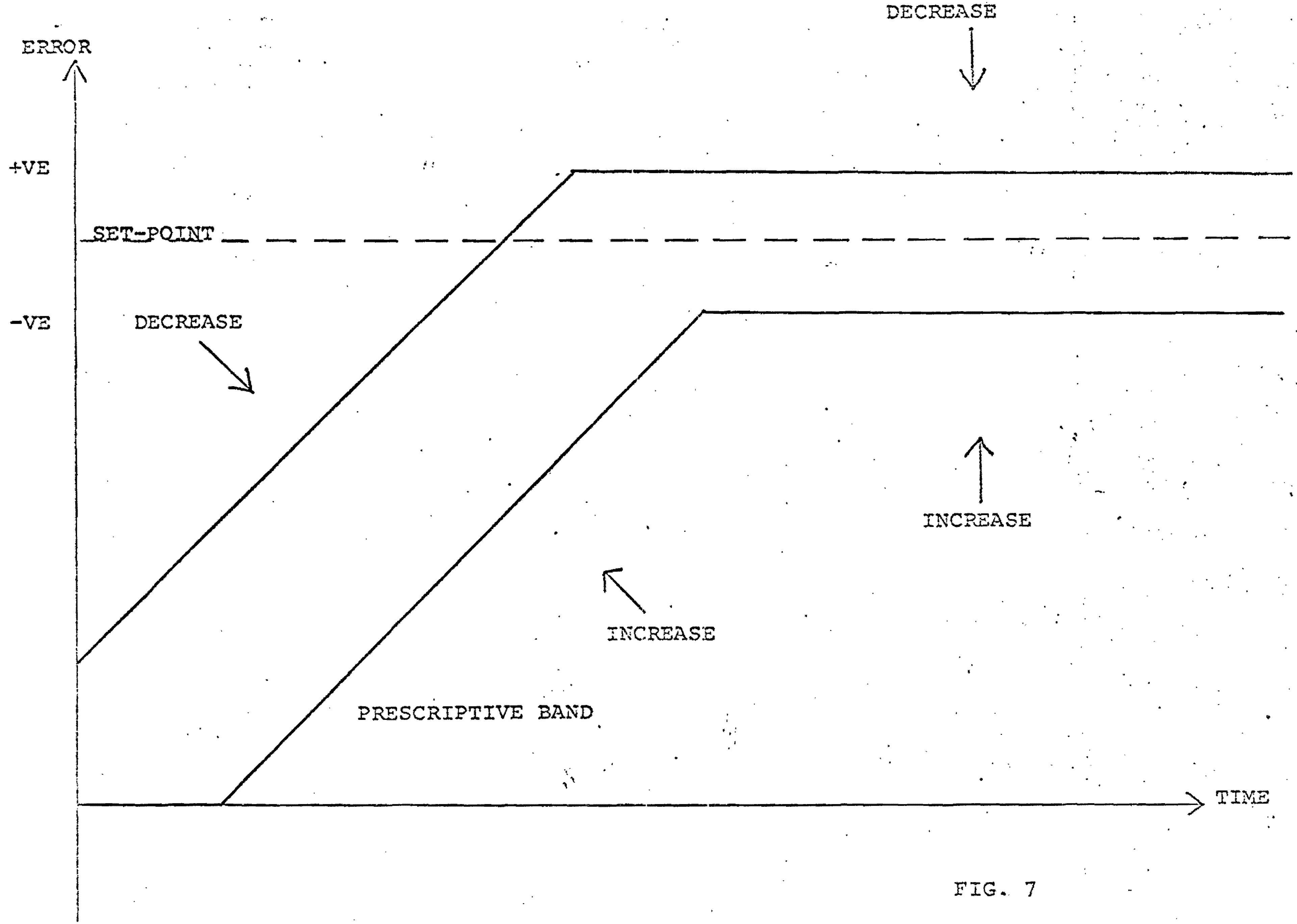
ERROR
+VE
SET-POINT
-VE
DECREASE
DECREASE
INCREASE
INCREASE
PRESCRIPTIVE BAND
TIME

FIG. 7

CHANGE IN ERROR \ ERROR	NB	NM	NS	NZ	PZ	PS	PM	PB
NB	PB*	PB*	NS	NM	PM	PS		NB*
NM	PM*		NS	NM	PM	PS		
NS	PM	PM	ZE	NS	PS	ZE	NM	NM
ZE	PB	PB	PM	ZE	ZE	NM	NB	NB
PS	PB	PB	PM	PS	NS	NM	NB	NB
PM			PB	PM	NM	NB		
PB	PB	PB	PB	PM	NM	NB	NB	NB

(A)

CHANGE IN ERROR \ ERROR	NB	NM	NS	NZ	PZ	PS	PM	PB
NB	PB	PB	NB	PB	PS	PB		
NM	PB		ZE	ZE	PB	NS		NB
NS	PS		PS	ZE	PM	NM		NB
ZE	PB		ZE	ZE	PS	NM		
PS		PS	PS	PB	ZE	NB		
PM			ZE	ZE	NM	NS		
PB	PB	PS	PB	PS	NS	NB	NB	NB

(B)

FIG.8

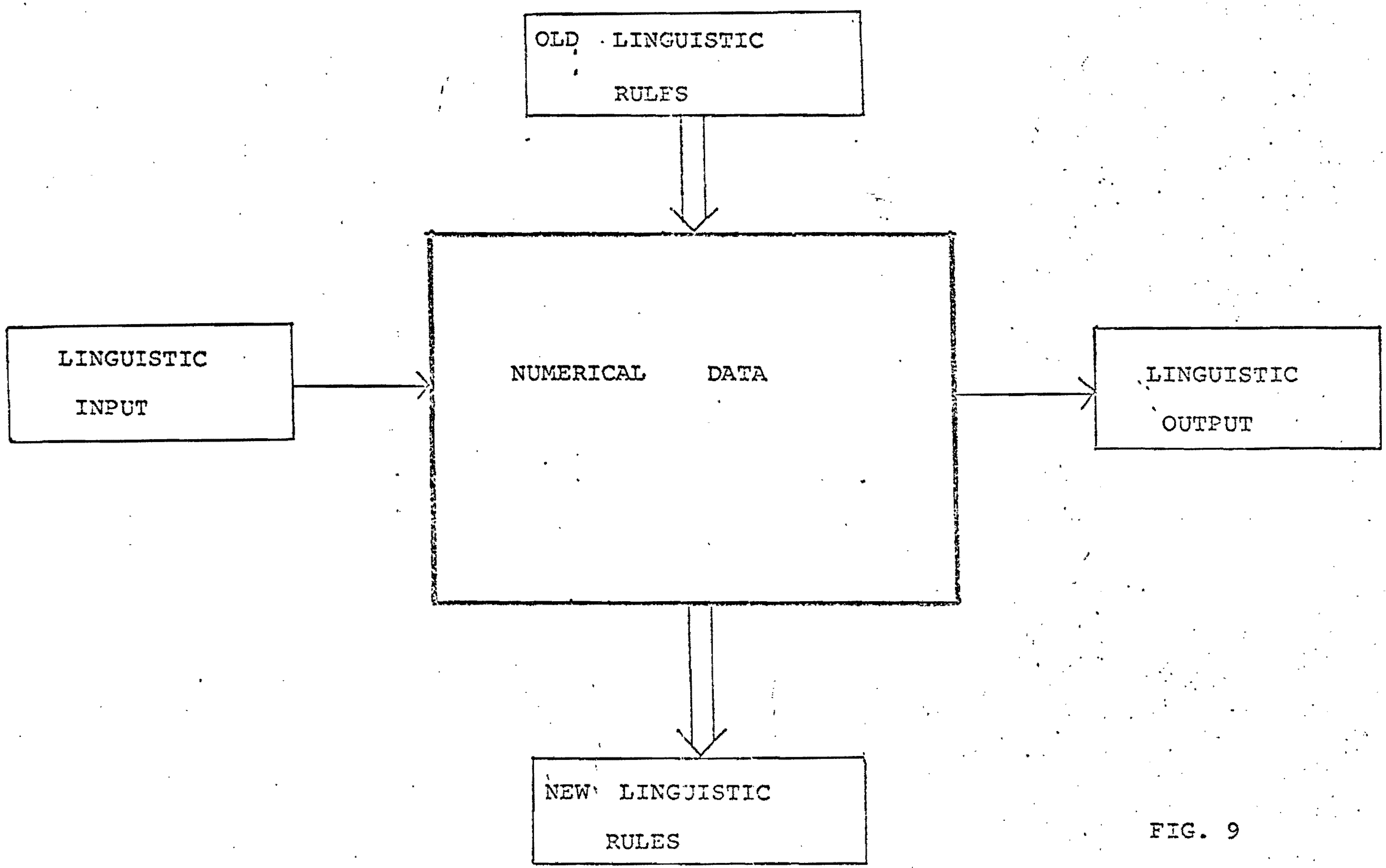
OLD LINGUISTIC RULES
LINGUISTIC INPUT
NUMERICAL DATA
LINGUISTIC OUTPUT
NEW LINGUISTIC RULES

FIG. 9

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A Fuzzy Logic Controller
for a Traffic Junction

by

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ABSTRACT

Work done on the implementation of a fuzzy logic controller in a single intersection of two one-way streets is presented. The model of the intersection is described and validated and the use of the theory of fuzzy sets in constructing a controller based on linguistic control instructions is introduced. The results obtained from the implementation of the fuzzy logic controller are tabulated against those corresponding to a conventional effective vehicle-actuated controller. With the performance criterion being the average delay of vehicles it is shown that the use of a fuzzy logic controller results in a better performance.

INTRODUCTION

A considerable amount of work has been done on the problem of modelling and controlling traffic junctions. Although the major problem in cities concerns sets of intersections (not individual ones) any approach to this problem should also include a sufficient description of the events occuring in any individual intersection in the linked or disjoint system under study.

Zadeh's pioneering work on fuzzy sets, by which a conceptual framework is provided for dealing with problems of vagueness in the representation of complex processes, can be of great help to the task of constructing a controller for such an individual traffic intersection. Indeed, the strength of the theory of fuzzy sets lies in its capability of rendering a powerful conceptual basis for the modelling and analysis of such processes, to which the human approach is characterised by rough approximations. Note that, although stochastic and fuzzy logics can both be regarded as derived from a probability logic [1], a stochastic approach would be methodologically different from the fuzzy discipline which has been used here. It seems, therefore, that the fuzzy rather than the stochastic approach should be used as the domain for the implementation of heuristics.

Previous work reported in the literature (e.g. [2], [3], [4], [5]) has shown the merits of the theory of fuzzy sets when applied to the design of controllers for real dynamic plants, industrial processes etc. In this study, the system is a traffic junction and the problem of its control is considered as a classical example of non-programmed decision making, i.e. decision making characterised by the lack of well specified analytical means for coping with a particular problem. Thus a linguistic control algorithm is synthesized capable of dealing with a continuously reproduced decision making situation. The starting point is an adequate (though qualitative) knowledge of the system and a protocol of control instructions used by a human operator. A fuzzy set theoretic representation of these instructions (which we call "a fuzzy logic controller") was tried as an answer to the control modelling problem, which gave very satisfactory results.

The work done on the construction of the model of the system and the implementation of the fuzzy logic controller is presented below. In order to validate the model a fixed-cycle controller was also simulated. The average delays of the vehicles resulted from the implementation of the fuzzy logic controller were compared to those caused by an efficient vehicle-actuated one. The results obtained show that the performance of the system is better under the fuzzy logic controller.

THE MODEL

The major assumption concerning the model is that the arrivals of vehicles at the intersection are considered as being random. Note that this assumption affects not only the truthfulness of the model but the selection of the control policies as well. The cycle is divided into two periods of "effective red" and "effective green" for each phase, the first corresponding to the halted traffic and the second to the traffic having the right of way. A total lost time of 10 seconds per cycle is assumed. Vehicles leave the queue at a constant rate equal to the saturation flow during the effective green (see [6], [7] for definitions). The saturation flow equals 3600 vehicles per hour at both arms. There is no turning traffic.

For each successive time unit a pseudo random number is generated and compared to some fixed quantity, which is equal to the mean rate of arrivals. Thus the arrival of a vehicle is decided. Let

$$q_n = \begin{cases} 1 \text{ if a vehicle arrived during the nth unit interval} \\ 0 \text{ otherwise} \end{cases}$$

If Q_G denotes the number of vehicles not cleared during the previous effective green period of a phase, then the queue Q_n at the nth time-unit after the beginning of the effective red of that phase would be:

$$Q_n = Q_G + \sum_{n_1=1}^{n} q_{n_1}$$

and the total waiting time of the vehicles in the queue would be:

$$D_{n,R} = \sum_{n_2=1}^{n} (Q_G + \sum_{n_1=1}^{n_2} q_{n_1})$$

Let s be the saturation flow, i.e. the rate at which vehicles are cleared during the effective green period. At the nth time unit after the beginning of the effective green, the number of vehicles not yet cleared would be:

$$Q_n = z.(Q_R + \sum_{n_1=1}^{n} q_{n_1} - s.n)$$

where Q_R is the queue which was built up during the previous effective red period of the phase, and z is equal to 1 when multiplied by a non-negative quantity and 0 otherwise.

These vehicles have been subjected to a delay:-

$$D_{n,G} = \sum_{n_1=1}^{n} z\cdot(Q_R + \sum_{n_2=1}^{n_1} q_{n_2} - s\cdot n_1)$$

Thus during a cycle, the total delay experienced by vehicles travelling along one arm of the intersection would be:-

$$D = D_{R,R} + D_{G,G}$$

where $D_{R,R}$, $D_{G,G}$ are the delays during R and G, i.e. the whole effective red and green periods respectively.
Finally the average delay per vehicle would be:-

$$d = \frac{D}{\sum_{n=1}^{R+G} q_n}$$

The model just described is quite simple in comparison to some more sophisticated ones (see, for example, [8], [9]) yet it suffices for the purpose of this work. A measure of its reliability was obtained by using a fixed-cycle controller which was implemented to the system. The system was subjected to a wide range of averages of random vehicles arrivals. Each time it was run for 7200 simulated seconds and the corresponding average delay per vehicle was calculated. Results of the calculations, together with the expected average delays, are given in Table 1. The expected ones have been obtained from the following formula (see [6]):

$$d = \frac{C(1-\lambda)^2}{2(1-\lambda X)} + \frac{X^2}{2_q(1-X)} - 0.65\left(\frac{C}{q^2}\right)^{1/3} X^{(2+5\lambda)}$$

where d = average delay per vehicle on the particular arm

C = cycle time

λ = proportion of the cycle which is effective green for the phase under consideration ($=g/c$)

q = flow

s = saturation flow

X = degree of saturation ($q/\lambda s$)

This formula gives the average delay of vehicles arriving at random at an intersection controlled by a fixed cycle controller. Its first two terms have a theoretical meaning while the last one is purely empirical. Table 1 shows a fair agreement between the calculated delays and those obtained from the above formula, thus providing a validation of the model.

The results of Table 1 actually correspond to optimum settings, i.e. optimum cycle and effective green times for the respective flow rates according to:

$$c_o = \frac{1.5L + 5}{1 - Y}$$

$$g_1 = \frac{y_1}{Y}(c_o - L)$$

$$g_2 = \frac{y_2}{Y}(c_o - L)$$

where c_o = optimum cycle time

g = effective green time

y = q/s

Y = $y_1 + y_2$

L = total lost time per cycle (10 seconds)

It should be noted that the delays of Table 1 correspond to random arrivals having fixed averages. In other words, should the average rates of arrivals be different from those for which optimum settings have been found and used for the control of the intersection, the resulting delays would also be different from those of Table 1. The use of timers, for tuning the controller in order to adjust its settings to the daily flow pattern, would not be an easy task. Consequently the actual delays occuring in any intersection controlled by fixed-cycle controllers would be by far in excess of those shown in Table 1, especially in cases of heavy traffic.

In the case of vehicle actuated controllers, the results of Table 1 correspond to those which should be expected with an efficient vehicle actuated installation [7]. That is, a vehicle actuated controller with speed timing or with a low fixed extention operation would result in delays as those of Table 1 for the respective flow rates. These delays were the basis for the comparison between vehicle actuated controllers and the fuzzy logic one (a fixed-cycle controller for a single intersection is scarcely worthy of comparison).

The system control process is shown in Figure 1. The intervention of the controller takes place every 10 seconds during each phase's effective green period; the first intervention taking place just after the first 7 seconds of the period. At each intervention the length of the extension of the effective green time for the phase having the right of way is decided. Information concerning the flow pattern is collected by detection pads, which, it is assumed, have been installed before the traffic lights in both arms of the intersection. The role of the detection pads is very important, as will be made clear in the sequel. It was assumed for the calculations that the flow pattern, as detected at the pads, is preserved throughout the period after each intervention for the phase having the right of way. The distance between the pads and the stop lines is sufficient for the controller to be informed about the arrivals of vehicles in both arms of the intersection during the next 11½ seconds, assuming that the effective green ends at the middle of the 3 seconds amber period.

Thus vehicle i passing over the detectors is registered in the following way: Its speed v_i is calculated. Assuming that its speed is preserved constant during its trip from the detectors to the junction, vehicle i will be at the "critical point" in (L/v_i-1.5 secs) time (see Figure 2). The "critical point" is the point where, should the lights turn to amber, it would be possible for the vehicle just to pass. Let

$$N_i = L/v_i - 1.5 \text{ secs}$$

be the number of seconds required for the vehicle to arrive at the critical point. N_i indicates the position of vehicle i in the flow pattern array for the next 10 seconds interval. The control input parameters are two continuously updated arrays corresponding to the halted traffic and the traffic having the right of way.

THE FUZZY LOGIC CONTROLLER

In order to make our exposition self-contained, some of the basic definitions of the theory of fuzzy sets([10], [11]) which were used to model the control algorithm, are given below.

A fuzzy set F of a Universe of Discourse U={x} is defined as a mapping $\mu_F(x):U\rightarrow[0,1]$ by which each x is assigned a number in

[0,1] indicating the extent to which x has the attribute F. Thus, if x is the number of vehicles in a queue, then "small" may be considered as a particular value of the fuzzy variable "queue" and each x is assigned a number $\mu_{SMALL}(x) \in [0,1]$ which indicates the extent to which that x is considered to be small.

Given the fuzzy sets A, B or U, the basic operations on A, B are:

(i) The complement $\bar{A}$ of A, defined by

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

(ii) The union AUB of A and B, defined by

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

(iii) The intersection A∩B of A and B, defined by

$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

A fuzzy relation R from U={x} to V={y} is a fuzzy set on the Cartesian product UxV, characterised by a function $\mu_R(x,y)$, by which each pair (x,y) is assigned a number in [0,1] indicating the extent to which the relation R is true for (x,y). There are several ways of constructing $\mu_R(x,y)$. The one used here will be seen later.

Finally given a fuzzy relation R from U to V and a fuzzy set A on U, a fuzzy set B on V is induced, given by the compositional rule of inference:

$$B = A^{\circ}R$$

or

$$\mu_B(y) = \max_x \{\min\{\mu_R(x,y), \mu_A(x)\}\}$$

A heuristic approach to the control problem was employed, which resulted in a set of linguistic control statements. The above basic ideas of the theory of fuzzy sets were used for the quantitative interpretation of these instructions as well as the decision-making process.

The fuzzy control instructions (see Appendix for a complete set) are of the form:

If T = medium
  and A = mt(medium)
  and Q = lt(small)
Then E = medium

Else

If T = long
  and A = mt(many)
  and Q = lt(medium)
Then E = long

Else etc.

where T = the fuzzy variable "time", which is assigned values like "very short", "short", "medium" etc.

A = the fuzzy variable "arrivals" i.e. the number of vehicles arriving at the arm having the right of way, which may be assigned values like "many", "more than a few" etc.

Q = the fuzzy variable "queue", which is assigned values like "any", "less than small" etc.

E = the fuzzy variable "extension", which is identical to "time"

The terms "medium", "more than medium", "less than small" etc. are labels of fuzzy sets defined on the relevant universes of discourse T,A,Q,E. Tables 2, 3 and 4 show the fuzzy sets used in this application. Further to the above basic operations, in this application we have introduced the operators "mt" and "lt", standing for "more than" and "less than" respectively. These are defined as follows: if A is a fuzzy set defined on $U = \{x_i\}$, $\mu_A(x_i)$ is its grade of membership function and x_o is the element of U for which

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$\mu_A(x_i)$ is maximum, then lt(A) and mt(A) are fuzzy sets defined on U such that:

$$\mu_{lt(A)}(x_i) = 0 \text{ for } x_i \geq x_o$$

$$= 1 - \mu_A(x_i) \text{ for } x_i < x_o$$

$$\mu_{mt(A)}(x_i) = 0 \text{ for } x_i \leq x_o$$

$$= 1 - \mu_A(x_i) \text{ for } x_i > x_o$$

The result of these operations on the fuzzy sets of Tables 3 and 4 above is shown in Tables 5 and 6.

Obviously

lt(A) or mt(A) = not(A)

lt(A) and mt(A) = 0

or

$$\max\{\mu_{lt(A)}(x_i), \mu_{mt(A)}(x_i)\} = 1 - \mu_A(x_i)$$

$$\min\{\mu_{lt(A)}(x_i), \mu_{mt(A)}(x_i)\} = 0$$

Note that if a fuzzy assignment like "A = small" is characterised by the poor content of the information conveyed, a fuzzy assignment like "A = less than small" is conveying even less information. In other words, fuzzy assignments like "A = less than small" are used whenever the grade of fuzziness is high.

"Any" is considered as a fuzzy set with all the elements of its universe of discourse been assigned a grade of membership equal to 1.

A total of 25 rules were used (5 for each intervention). Every rule is a fuzzy relation between the inputs T, A, Q. and the output E. The connectives "and" and "else" are interpreted as the operators "min" and "max" respectively. Thus:-

T = very short

and A = mt(none)

and -

is a fuzzy phrase P (see [12])defined on the universe of discourse TxAxQ with grades of membership function

$$\mu_P(t,a,q)=\min\{\mu_{v.short}(t),\mu_{mt(none)}(a),\mu_{any}(q)\}$$

The fuzzy implication "if P then E = very short" is also a fuzzy phrase R defined on TxAxQxE with grades of membership function

$$\mu_R(t,a,q,e)=\min\{\mu_{v.short}(t),\mu_{mt(none)}(a)\,\mu_{any}(q),\mu_{v.short}(e)\}$$

Finally two or more fuzzy implications R,S...., connected by "else" form a fuzzy clause C defined on TxAxQxE with grades of membership function

$$\mu_C(t,a,q,e)=\max\{\mu_R(t,a,q,e),\mu_S(t,a,q,e),...\}$$

In this application, since each fuzzy rule is represented by a 4-dimensional matrix, the fuzzy algorithm employed at each intervention for deciding the control action is represented by the union of 5 such matrices, as five rules operate at each intervention. 25(5x5) rules given in the appendix thus provide for a maximum of 5 interventions (each consisting of 5 rules) taking place at 7th, 17th, 27th, 37th and 47th second. Thus the maximum possible effective green time is 57 seconds. At each intervention the 5 rules are invoked in the manner described below 10 times (i.e. for each of the next 10 seconds). Note that, as the detecting pads are sufficiently far away from the junction, at each intervention, data is available for each of the next 10 seconds. Consider now the (t_i,a_j,a_k,e_l) entry of the matrix C_2, corresponding to the algorithm used at the 2nd intervention of the controller, where:

$t_i = 8$ (i.e. we consider the next 8 seconds)

$a_j = 4$ (i.e. 4 vehicles will cross the critical point if no change of the current state of the system occurs during the next 8 seconds)

$q_k = 5$ (i.e. 5 vehicles queue will build up if no change of the current state of the system occurs during the next 8 seconds).

$e_1 = 8$ (i.e. the extention given to the present state of the system is 8 seconds)

The first control statement R_1 for the second intervention (see appendix) is:

If T = very short
and A = mt(none)
and Q = any
then E = very short

From Tables 2, 5, 6 we have:

$$\mu_{v.short}(8) = 0.0$$

$$\mu_{mt(none)}(4) = 1.0$$

$$\mu_{any}(5) = 1.0$$

Thus

$$\mu_{R_1}(8,4,5,8) = \min\{\mu_{v.short}(8), \mu_{mt(none)}(4), \mu_{any}(5), \mu_{v.short}(8)\}$$

$$= \min\{0,1.0,1.0,0\} = 0$$

Similarly we find

$$\mu_{R_2}(8,4,5,8) = \min\{\mu_{short}(8),\mu_{mt(a\ few)}(4),$$
$$\mu_{lt(v.short)}(5),\mu_{short}(8)\}$$
$$= \min\{0,.9,.5,0\} = 0$$

$$\mu_{R_3}(8,4,5,8) = \min\{\mu_{medium}(8),\mu_{mt(few)}(4),$$
$$\mu_{lt(v.short)}(5),\mu_{medium}(8)\}$$
$$= \min\{0,.8,.5,0\} = 0$$

$$\mu_{R_4}(8,4,5,8) = \min\{\mu_{long}(8),\mu_{mt(medium)}(4),$$
$$\mu_{lt(v.short)}(5),\mu_{long}(8)\}$$
$$= \min\{.5,.5,.5,.5\} = .5$$

$$\mu_{R_5}(8,4,5,8) = \min\{\mu_{v.long}(8),\mu_{mt(many)}(4),$$
$$\mu_{lt(short)}(5),\mu_{v.long}(8)\}$$
$$= \min\{.5,0,1,.5\} = 0$$

Thus the (t_i,a_j,q_k,e_l) entry of matrix C_2 is

$$\mu_{C_2}(t_i,a_j,q_k,e_l) = \max\{0,0,0,.5,0\} = .5$$

THE PROCEDURE FOR DECIDING THE CONTROL ACTION

Having determined the entries of the matrix corresponding to the algorithm for each intervention, the process of inferring the control action is carried out as follows.

For each successive time unit (=1 second) for the next 10 secon[ds] data concerning vehicles crossing the critical point and vehicles added to the queue are used as inputs to the algorithm matrix in use. The corresponding entry of the matrix is thus determined. This entry is a measure of the confidence with which the

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algorithm may be applied, for the corresponding data. Obviously, that extension will be selected which corresponds to the maximum degree of confidence. In other words, fuzzy predictive decision making implies that, that action is selected which minimizes fuzziness. Thus, given a set of fuzzy rules choose the one which is provided for coping with conditions as similar to the actual ones as possible. And, given a set of alternative actual conditions, consider those which are as similar to the conditions, for which the algorithm provides, as possible.

The explicit description of the procedure for deciding the control action is given below, by means of an example. Thus we consider the controller's 2nd intervention. Arm N-S has the right of way. There are 5 vehicles queued at E-W arm. Data, concerning number of vehicles crossing the critical point (N-S traffic) and queued (E-W traffic) at each successive time unit during the next 10 seconds, is summarized in arrays α and α' respectively.

$$\alpha = (0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1)$$

$$\alpha' = (0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0)$$

From α, α' arrays β, β' are constructed

$$\beta = (0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 5 \quad 5 \quad 6)$$

$$\beta' = (5 \quad 6 \quad 6 \quad 6 \quad 7 \quad 7 \quad 7 \quad 8 \quad 8 \quad 8)$$

as follows: if the ith elements of α and β are a_i, b_i respectively, it is

$$b_i = \sum_{j=1}^{i} a_j$$

and if the ith elements of α' and β' are a_i', b_i' respectively, it is

$$b'_i = Q + \sum_{j=1}^{i} a'_j$$

where Q is the present queue at E-W arm. For example, if no change of the current state of the system occurs during the next 6 seconds, it is seen (from β and β') that 4 vehicles will cross the critical point and there will be a total of 7 vehicles in the queue at E-W arm after 6 seconds from now.

The ith elements of arrays β, β', i=1,...,10, determine the appropriate entry of the algorithm matrix C_2, which indicates the applicability of the algorithm to the situation described by these elements of the arrays. Thus, for t=1 second (i.e. considering an extension of 1 second) we have that no vehicle will cross the critical point (first element of array β) and that the queue will remain the same (5 vehicles, first element of array β'). It is easy to show that the rules of the algorithm are assigned the grade 0, and, consequently, the algorithm is assigned the grade 0 for t=1 second. The results have been summarized in Table 7. Obviously, the controller will select the extension of 10 seconds. Thus, the state of the system will remain the same for the next 10 seconds and the above procedure will be repeated (with new β and β') at the end of the 10 seconds period. If the extension given to the present state of the system were less than 10 seconds, the state of the system would change at the end of the extension period.

Note that if all the entries of the last row of Table 7 were less than 0.5, no extension would be given and the state of the system (i.e. the phase) would be immediately changed.

Finally, if the maximum of the entries of the last row were not unique, i.e. if two or more alternative extension periods were indicated, then the maximum of these alternative extension periods would be selected. Of course, some other rule might be used instead (e.g. one giving the minimum extension period or the median or one randomly selected among these alternatives). For this, however, another control algorithm would be required in the place of the one used in this implication, which was based on the rule giving the maximum extension period.

RESULTS AND FINAL REMARKS

Because of the random nature of the arrivals assumed, many runs of the model were needed, in order to get reliable results. The simulation work was carried out on the ICL-1900 general purpose computer.

The results have been summarised in Table 8, whence it is clear that the system's performance is best under the fuzzy logic controller for all possible combinations of flow rates. Note that the effectiveness of the controller, as indicated by the percentage improvement in delays, is not seriously affected by the total volume of traffic through the junction.

These results have been obtained after several modifications of the control instructions initially set. The trial and error method was used in order to obtain an effective set of rules (or what is termed "satisfying control" in Management Science). In other words, a learning procedure was employed, by which the human performance in a similar real life situation of controlling a traffic junction is derived.

It is interesting to note that, having defined (in terms of fuzzy sets) what was considered to be a "small queue" or "few arrivals", it was the rules rather than the fuzzy sets which were modified. It is quite apparent that if the parameters describing the membership functions were introduced as additional inputs to the decision concerning the control algorithm, the dimensionality of the problem would radically increase, thus imposing severe difficulties in the obtainment of its solution.

The question of stability has been part of the whole problem of obtaining what is termed "effective set of rules". In this particular application, the stability of the system is defined as the condition of the system not getting saturated if subjected to a wide range of flow rates.

As far as fuzzy set theory is concerned, its basic concept, "fuzziness", characterises only a state of knowledge. It exists neither in the system nor in the controller but in the human mind. Although the controller which was actually designed is termed "a fuzzy logic controller", it actually acts deterministically. That is, the algorithm by means of which the decision is taken, although conceived in fuzzy, linguistic terms, is not fuzzy after the actual design is completed, i.e. after the fuzzy sets and implications are established [13].

It is hoped that further work will be done on the problem of controlling traffic by use of the theory of fuzzy sets. It must however be kept in mind that the fuzzy logic controller was designed for the purpose of controlling traffic characterised by randomness. In a linked system the traffic would be modulated. This should be taken into account when considering a controller for an intersection being part of a whole network, forming an integrated control system. On the other hand special problems would arise in this case owing to the hierarchical structure of the system and consequently the control policy itself. It is thought also that in the case of integrated traffic control systems the theory of fuzzy sets would show its merits much more so than in the present simple case of an individual intersection.

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THE FUZZY ALGORITHM

INTERVENTION: 7nth second

```
        If       T=very short
           and   A=mt(none)
           and   Q=any
        then     E=very short
ELSE
        If       T=short
           and   A=mt(a few)
           and   Q=lt(very small)
        then     E=short
ELSE
        If       T=medium
           and   A=mt(few)
           and   Q=lt(very small)
        then     E=medium
ELSE
        If       T=long
           and   A=mt(medium)
           and   Q=lt(very small)
        then     E=long
ELSE
        If       T=very long
           and   A=mt(many)
           and   Q=lt(very small)
        then     E=very long
```

INTERVENTION:17nth second

```
        If      T=very short
            and A=mt(none)
            and Q=any
        then    E=very short
ELSE
        If      T=short
            and A=mt(a few)
            and Q=lt(very small)
        then    E=short
ELSE
        If      T=medium
            and A=mt(few)
            and Q=lt(very small)
        then    E=medium
ELSE
        If      T=long
            and A=mt(medium)
            and Q=lt(very small)
        then    E=long
ELSE
        If      T=very long
            and A=mt(many)
            and Q=lt(small)
        then    E=very long
```

INTERVENTION: 27nth second

```
        If      T=very short
            and A=mt(none)
            and Q=any
```

then E=very short

ELSE

If T=short

and A=mt(a few)

and Q=lt(very small)

then E=short

ELSE

If T=medium

and A=mt(few)

and Q=lt(very small)

then E=medium

ELSE

If T=long

and A=mt(medium)

and Q=lt(very small)

then E=long

ELSE

If T=very long

and A=mt(many)

and Q=lt(small)

then E=very long

INTERVENTION: 37nth second

If T=very short

and A=mt(none)

and Q=any

then E=very short

ELSE

If T=short

and A=mt(a few)

and Q=lt(small plus)

```
            then        E=short
ELSE
            If          T=medium
                 and    A=mt(medium)
                 and    Q=lt(small plus)
            then        E=medium
ELSE
            If          T=long
                 and    A=mt(many)
                 and    Q=lt(medium)
            then        E=long
ELSE
            If          T=very long
                 and    A=mt(too many)
                 and    Q=lt(long
            then        E=very long
INTERVENTION: 47nth second
            If          T=very short
                 and    A=mt(none)
                 and    Q=any
            then        E=very short
ELSE
            If          T=short
                 and    A=mt(a few)
                 and    Q=lt(long)
            then        E=short
ELSE
            If          T=medium
                 and    A=mt(medium)
                 and    Q=lt(long)
```

```
        then        E=medium
ELSE
        If          T=long
            and     A=mt(too many)
            and     Q=lt(very long)
        then        E=long
ELSE
        If          T=very long
            and     A=mt(too many)
            and     Q=lt(very long)
        then        E=very long
```

TABLE 1

Average delays with fixed-cycle controller
and optimum settings

N - S traffic (veh/hr)	E - W traffic (veh/hr)	Delay (secs/veh)		Error %
		Model	Formula	
360	360	7.2	7.4	-.2
360	720	7.4	7.9	- 6
360	1080	7.9	8.4	- 5
360	1440	8.4	9.0	- 7
360	1800	9.3	10.2	- 9
360	2160	12.3	12.9	- 5
360	2520	15.8	18.9	-17
720	720	9.7	10.0	- 3
720	1080	10.8	11.6	- 7
720	1440	12.7	13.8	- 8
720	1800	15.9	17.3	- 8
720	2160	21.8	24.9	-12
1080	1080	13.6	14.9	- 9
1080	1440	17.9	19.7	- 9
1080	1800	25.8	29.2	-11
1440	1440	27.3	30.7	-11

TABLE 2

Fuzzy sets defined on Time (or Extension)

Fuzzy sets \ time (secs)	1	2	3	4	5	6	7	8	9	10
very short	1	.5	0	0	0	0	0	0	0	0
short	0	.5	1	.5	0	0	0	0	0	0
medium	0	0	0	.5	1	.5	0	0	0	0
long	0	0	0	0	0	.5	1	.5	0	0
very long	0	0	0	0	0	0	0	.5	1	1

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TABLE 3

Fuzzy sets defined on Arrivals

F. sets \ Arrivals (veh)	1	2	3	4	5	6	7	8	9	10
none	.5	.2	.1	0	0	0	0	0	0	0
a few	1	.5	.2	.1	0	0	0	0	0	0
few	.5	1	.5	.2	.1	0	0	0	0	0
medium	.2	.5	1	.5	.2	.1	0	0	0	0
many	.1	.2	.5	1	.5	.2	.1	0	0	0
too many	0	.1	.2	.5	1	.5	.2	.1	0	0

TABLE 4

Fuzzy sets defined on Queues

queue (veh) / f.sets	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
very small	0	.5	.7	.9	.1	.9	.7	.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
small	0	0	0	0	0	.5	.7	.9	1	.9	.7	.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
small plus	0	0	0	0	0	0	0	0	0	.5	.7	.9	1	.9	.7	.5	0	0	0	0	0	0	0	0	0	0	0	0	0
medium	0	0	0	0	0	0	0	0	0	0	0	0	0	.5	.7	.9	1	.9	.7	.5	0	0	0	0	0	0	0	0	0
long	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.5	.7	.9	1	.9	.7	.5	0	0	0	0	0
very long	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	.5	.7	.9	1	.9	.7	.5	0

TABLE 5

"more than"-operation on the fuzzy sets of Table 3

Arrivals (vehicles) / fuzzy sets	1	2	3	4	5	6	7	8	9	10
mt(none)	.5	.8	.9	1	1	1	1	1	1	1
mt(a few)	0	.5	.8	.9	1	1	1	1	1	1
mt(few)	0	0	.5	.8	.9	1	1	1	1	1
mt(medium)	0	0	0	.5	.8	.9	1	1	1	1
mt(many)	0	0	0	0	.5	.8	.9	1	1	1
mt(too many)	0	0	0	0	0	.5	.8	.9	1	1

TABLE 6

"Less than" - operation on the fuzzy sets of Table 4

queue (veh) / f. sets	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
lt(very small)	1	.5	.3	.1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
lt(small)	1	1	1	1	1	.5	.3	.1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
lt(small plus)	1	1	1	1	1	1	1	1	1	.5	.3	.1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
lt(medium)	1	1	1	1	1	1	1	1	1	1	1	1	1	.5	.3	.1	0	0	0	0	0	0	0	0	0	0	0	0	0
lt(long)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	.5	.3	.1	0	0	0	0	0	0	0	0	0
lt(very long)	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	.5	.3	.1	0	0	0	0	0

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TABLE 7

Decision Table for the control action

Fuzzy Control Statement			Time (seconds)									
time	passing vehicles	queue	1	2	3	4	5	6	7	8	9	10
very short	mt(none)	any	0	.5	0	0	0	0	0	0	0	0
short	mt(a few)	lt(very short)	0	0	0	.3	0	0	0	0	0	0
medium	mt(few)	lt(very short)	0	0	0	0	.1	.1	0	0	0	0
long	mt(medium)	lt(very short)	0	0	0	0	0	.1	.1	0	0	0
very long	mt(many)	lt(short)	0	0	0	0	0	0	0	.5	.5	.8
Fuzzy Algorithm			0	.5	0	.3	.1	.1	.1	.5	.5	.8

TABLE 8

Comparison between delays caused by an efficient vehicle-actuated controller and the fuzzy-logic one

N - S traffic (veh/hr)	E - W traffic (veh/hr)	Average overall delay (secs/veh)		Improvement %
		Vehicle-actuated controller	Fuzzy-logic controller	
360	360	7.2	5.7	+21
360	720	7.4	6.1	+18
360	1080	7.9	6.6	+17
360	1440	8.4	7.3	+13
360	1800	9.3	8.4	+10
360	2160	12.3	10.0	+19
360	2520	15.8	13.6	+14
720	720	9.7	7.4	+21
720	1080	10.8	8.8	+19
720	1440	12.7	10.9	+14
720	1800	15.9	14.1	+11
720	2160	21.8	18.5	+15
1080	1080	13.6	12.0	+12
1080	1440	17.9	15.4	+14
1080	1800	25.8	21.6	+16
1440	1440	27.3	22.9	+16

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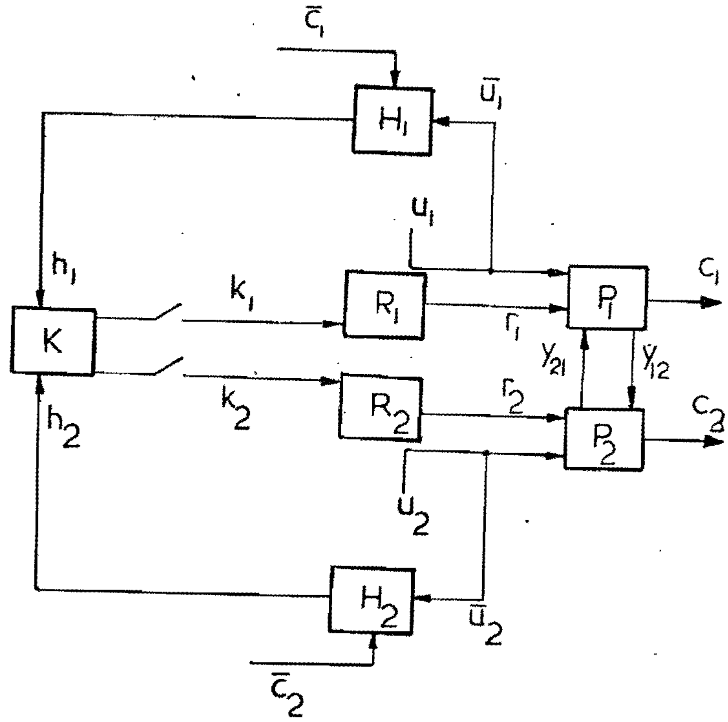

Figure 1. The system control process

P_i	processing of vehicles within the junction along route i
R_i	regulating function (signal setting)
H_i	data processing
K	optimizing function (control algorithm)
u_i	vehicles on route i entering the junction
c_i	" " " leaving " "
y_{ij}	subprocess's P_i interaction with subprocess P_j
$\bar{c}_i$	premeasured data concerning c_i (saturation flow).

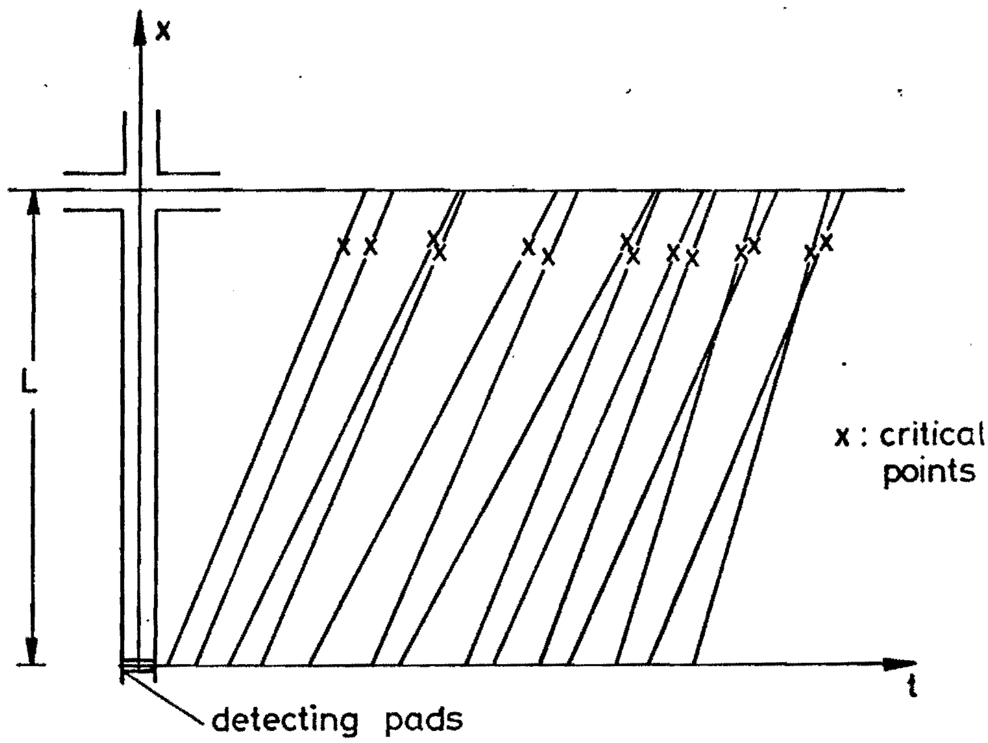

FIGURE 2. Time-space diagram.

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FUZZY RELATIONAL EQUATIONS AND THE INVERSE PROBLEM

by

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SUMMARY

The inverse problem concerned with fuzzy relations is investigated. The conditions for the existence of a solution are shown and an analytical solution is given. A method for the improvement of the solution is proposed.

1. INTRODUCTION

This paper is related to Sanchez's (1976) work on fuzzy relational equations. He dealt with the problem "Given two fuzzy relations $Q \subset U \times V$ and $S \subset U \times W$, find $R \subset V \times W$ such that $R \circ Q = S$", where $\circ$ denotes maxmin composition. He showed an existence condition of the solutions by giving the least upper bound of the solutions. In general, the set of all the possible solutions for the above equation forms an upper semi lattice. Therefore, the greatest lower bound does not always exist.

The paper discusses the problem called the "inverse problem", "Given a fuzzy relation $R \subset U \times V$ and a fuzzy subset $B \subset V$, find all $A \subset U$ such that $A \circ R = B$". Although it is a special form of Sanchez equation, this fuzzy relational equation is widely used because of its simplicity and it is also very useful in practical applications (see Zadeh (1973), Mamdani and Assilian (1975), Pappis and Mamdani (1976)). For example, a set of fuzzy implications (or fuzzy conditional statements) of the form "If A_i then B_i, $i \varepsilon I$" can be conveniently expressed by the union of Cartesian products $R = \bigcup_{i \varepsilon I} A_i \times B_i$, $A_i \subset U$, $B_i \subset V$, $i \varepsilon I$. Given $A \subset U$, then $B \subset V$ is induced, according to the fuzzy relational equation $A \circ R = B$.

The paper gives a different type of the existence conditions of the solutions, which is related to the lower bounds of the solutions. The lower bounds are analytically obtained by the method presented in the paper. Thus, when a fuzzy system is described by a relation matrix R associated with the maxmin compositional rule, the set of all the possible inputs A's, which give the same output B, can be obtained by combining the least upperbound and a number of the lower bounds.

2. STATEMENT OF THE PROBLEM

Denote a fuzzy subset A of $U = \{u_i/i=1,\ldots,m\}$ by

$$A = \{(u_i,a_i)/i=1,\ldots,m\},$$

a fuzzy subset B of $V = \{v_j/j=1,\ldots,n\}$ by

$$B = \{(v_j,b_j)/j=1,\ldots,n\},$$

and a fuzzy relation R of U x V by

$$R = \{((u_i,v_j),r_{ij})/i=1,\ldots,m,\ j=1,\ldots,n\}$$

where a_i, b_j and r_{ij} are the grades of membership of u_i, v_j and (u_i, v_j), respectively.

The composition of A with R, denoted by $A \circ R$, is defined to be a fuzzy subset B associated with the grades of membership

$$b_j = \bigvee_i (a_i \wedge r_{ij}),\ 1 \leq j \leq n.$$

Our problem can be stated as follows:

"Given R and B, find all A's such that $A \circ R = B$".

3. EXISTENCE OF A SOLUTION

3.1 Notations

Let row vectors $\underline{a} = (a_1 a_2 \ldots a_m)$, $\underline{b} = (b_1 b_2 \ldots b_m)$, $\underline{c} = (c_1 c_2 \ldots c_n)$ and the mxn matrices $\underline{R} = [r_{ij}]$, $\underline{S} = [s_{ij}]$. The following notations will be used:

$\underline{a} \geq \underline{b}$: $a_i \geq b_i,\ \forall_i$

$\underline{a} \leq \underline{b}$: $a_i \leq b_i,\ \forall_i$

$\underline{a} \wedge \underline{b}$: $(a_1 \wedge b_1\ a_2 \wedge b_2 \ldots a_m \wedge b_m)$

$\underline{a} \vee \underline{b}$	:	$(a_1 \vee b_1 \;\; a_2 \vee b_2 \ldots a_m \vee b_m)$
$\underline{a}=0$	:	$a_i = 0, \forall i$
$\hat{\underline{a}}$	:	$\bigvee_i (a_i)$
$\underline{R} \geq \underline{S}$	:	$r_{ij} \geq s_{ij}, \forall i, \forall j$
$\underline{R} \leq \underline{S}$	:	$r_{ij} \leq s_{ij}, \forall i, \forall j$
$\underline{a}^T$	:	transpose of $\underline{a}$
$\underline{R}^T$	:	transpose of $\underline{R}$
$\wedge \underline{R}$	:	$\underline{r}_1 \wedge \underline{r}_2 \wedge \ldots \wedge \underline{r}_m$ } where $\underline{r}_i$ is the ith row vector of $\underline{R}$
$\vee \underline{R}$	:	$\underline{r}_1 \vee \underline{r}_2 \vee \ldots \vee \underline{r}_m$

3.2 Definitions

In the sequel, small letters x, y, etc. are used to denote scalars and when underlined, like $\underline{a}$, $\underline{b}$ etc., they denote vectors. Capital letters R, S etc. are used to denote fuzzy subsets and when underlined, like $\underline{R}$, $\underline{S}$ etc., they denote matrices.

Any scalar and any elements of vectors or matrices are assumed to have their values in the interval [0,1].

o-composition

The o-composition of a vector $\underline{a} = (a_1 \; a_2 \ldots a_m)$ with a column vector $b = (b_1 \; b_2 \ldots b_m)^T$, denoted by $\underline{a} \circ \underline{b}$, is defined by the scalar

$$\underline{a} \circ \underline{b} \triangleq \bigvee_i (a_i \wedge b_i).$$

The o-composition of a row vector $\underline{a} = (a_1 \; a_2 \ldots a_m)$ with a mxn matrix $\underline{R} = [r_{ij}]$, denoted by $\underline{a} \circ \underline{R}$, is defined by the row vector

$$\underline{a} \circ \underline{R} \triangleq (\underline{a} \circ \underline{r}_1 \;\; \underline{a} \circ \underline{r}_2 \ldots \underline{a} \circ \underline{r}_n)$$

where $\underline{r}_j$ is the jth column vector of $\underline{R}$.

α-composition

The α-composition of a scalar x with a scalar y, denoted by $x\alpha y$, is defined by the scalar

$$x\alpha y \triangleq \begin{cases} 1 & \text{if } x \geqslant y \\ y & \text{if } x < y \end{cases}$$

Given a column vector $\underline{a} = (a_1\ a_2 \ldots a_m)^T$ and a scalar x, the α-composition of $\underline{a}$ with x, denoted by $\underline{a}\alpha x$, is defined by the column vector

$$\underline{a}\alpha x \triangleq (a_1\alpha x\ \ a_2\alpha x \ldots a_m\alpha x)^T.$$

Given a mxn matrix $\underline{R} = [r_{ij}]$ and a row vector $\underline{a} = (a_1\ a_2 \ldots a_n)$, let $\underline{r}_j$ be the jth column vector of $\underline{R}$. Then the α-composition of $\underline{R}$ with $\underline{a}$, denoted by $\underline{R}\alpha\underline{a}$, is defined by the mxn matrix

$$\underline{R}\alpha\underline{a} \triangleq [\underline{r}_1\alpha a_1\ \ \underline{r}_2\alpha a_2 \ldots \underline{r}_n\alpha a_n].$$

Thus $\underline{R}\alpha\underline{a}$ is a matrix, $\underline{R}\alpha\underline{a} = [w_{ij}]$, where

$$w_{ij} = r_{ij}\alpha a_j.$$

β-composition

The β-composition of a scalar x with a scalar y, denoted by $x\beta y$, is defined by the scalar

$$x\beta y \triangleq \begin{cases} 0 & \text{if } x < y \\ y & \text{if } x \geqslant y. \end{cases}$$

Given a column vector $\underline{a} = (a_1\ a_2 \ldots a_m)^T$ and a scalar x, the β-composition of $\underline{a}$ with x, denoted by $\underline{a}\beta x$, is defined by the column vector

$$\underline{a}\beta x \triangleq (a_1\beta x \; a_2\beta x \ldots a_m\beta x)^T.$$

Given a mxn matrix $\underline{R} = [r_{ij}]$ and a row vector $\underline{a} = (a_1 \; a_2 \ldots a_n)$, let $\underline{r}_j$ be the jth column vector of $\underline{R}$. Then the β-composition of $\underline{R}$ with $\underline{a}$, denoted by $\underline{R}\beta\underline{\alpha}$, is defined by the mxn matrix

$$\underline{R}\beta\underline{a} \triangleq [\underline{r}_1\beta a_1 \; \underline{r}_2\beta a_2 \ldots \underline{r}_n\beta a_n].$$

Thus $\underline{R}\beta\underline{a}$ is a matrix, $\underline{R}\beta\underline{a} = [z_{ij}]$,

where

$$z_{ij} = r_{ij}\beta a_j.$$

<u>Φ-sets</u>

Given a column vector $\underline{a} = (a_1 \; a_2 \ldots a_m)^T$, such that $a_i = \underline{\hat{a}}$ or 0, i=1,...,m, the set $\Phi(\underline{a})$ of column vectors $\phi(\underline{a})$ is defined as follows:

$$\Phi(\underline{a}) \triangleq \{\phi(\underline{a})\},$$

where

$$\phi(\underline{a}) = (\phi_1 \; \phi_2 \ldots \phi_m)^T,$$

$$\phi_i = 0 \text{ or } \underline{\hat{a}}, \; i=1,\ldots,m,$$

$$\sum_{i=1}^{m} \phi_i = \underline{\hat{a}}.$$

Thus, if there are k nonzero elements in $\underline{a}$, there are k vectors in $\Phi(\underline{a})$. Note that $\Phi(\underline{a})$ is defined iff $a_i = 0$ or $\underline{\hat{a}}$, $\forall i$.

Example: Let $\underline{a} = (0 \ \ .3 \ \ 0 \ \ 0 \ \ .3)^T$. Then $\hat{\underline{a}} = .3$ and $a_i = .3$ or $0, \forall i$, thus $\Phi(\underline{a})$ is defined and we have:

$$\Phi(\underline{a}) = \{(0 \ \ .3 \ \ 0 \ \ 0 \ \ 0)^T, (0 \ \ 0 \ \ 0 \ \ 0 \ \ .3)^T\}.$$

Given a mxn matrix $\underline{R} = [r_{ij}]$, let $\underline{r}_j$ be its jth column vector and assume that $\Phi(\underline{r}_j)$ is defined for $j=1,\ldots,n$. Then the set $\Phi(\underline{R})$ of matrices $\phi(\underline{R})$ is defined as follows:

$$\Phi(\underline{R}) \triangleq \{\phi(\underline{R})\},$$

where

$$\phi(\underline{R}) = [\,\phi(\underline{r}_1) \ \ \phi(\underline{r}_2) \ldots \phi(\underline{r}_n)\,].$$

Example: Let

$$\underline{R} = \begin{bmatrix} 0 & .8 & .5 \\ .2 & 0 & 0 \\ .2 & 0 & .5 \end{bmatrix}$$

We have

$$\Phi(\underline{R}) = \left\{ \begin{bmatrix} 0 & .8 & .5 \\ .2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & .8 & 0 \\ .2 & 0 & 0 \\ 0 & 0 & .5 \end{bmatrix}, \begin{bmatrix} 0 & .8 & .5 \\ 0 & 0 & 0 \\ .2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & .8 & 0 \\ 0 & 0 & 0 \\ .2 & 0 & .5 \end{bmatrix} \right\}.$$

Note that there are z matrices in $\Phi(\underline{R})$,

$$z = \prod_{j=1}^{n} z_j$$

where

$$z_j = \begin{cases} \text{number of nonzero elements in } \underline{r}_j, \text{ if } \underline{r}_j \neq 0 \\ 1, \text{ if } \underline{r}_j = 0. \end{cases}$$

3.3 Some properties of o, α, β compositions and Φ-sets

Given scalars x, y we have

P1 $x\alpha y \geq y$

P2 $x\beta y \leq y$

P3 $x\alpha y = y\alpha x \Leftrightarrow x=y$

P4 $x\beta y = y\beta x \Leftrightarrow x=y$

Given a column vector $\underline{b} = (b_1\ b_2 \ldots b_m)^T$ and a scalar x, we have:

P5 $\underline{b}\alpha x \geq \underline{b}\beta x$

P6 $\exists\ \underline{a}:\underline{a}\circ\underline{b} = x \Leftrightarrow \exists b_i \varepsilon \underline{b}: b_i \geq x$

Given a mxn matrix $\underline{R}$ and a row vector $\underline{b} = (b_1\ b_2 \ldots b_n)$ we have:

P7 $\underline{R}\alpha\underline{b} \geq \underline{R}\beta\underline{b}$.

Given $\underline{R}$, $\underline{b}$ as above and a row vector $\underline{a} = (a_1\ a_2 \ldots a_m)$ we have:

P8 $\underline{a}\circ\underline{R} = \underline{b} \Leftrightarrow \underline{a}\circ\underline{r}_j = b_j, \forall j$ ($\underline{r}_j$: jth column vector of $\underline{R}$).

3.4 The necessary and sufficient conditions

Lemma 1 Given a column vector $\underline{b} = (b_1\ b_2 \ldots b_m)^T$ and a scalar x, we have

$$\exists\ \underline{a}:\underline{a}\circ\underline{b} = x \Leftrightarrow (\underline{b}\alpha x)^T \circ \underline{b} = x \Leftrightarrow (\underline{b}\beta x)^T \circ \underline{b} = x \Leftrightarrow$$
$$(\phi(\underline{b}\beta x))^T \circ \underline{b} = x, \forall \phi(\underline{b}\beta x) \varepsilon \Phi(\underline{b}\beta x).$$

<u>Lemma 2</u> Given a row vector $\underline{a} = (a_1\ a_2 \ldots a_m)$, a column vector $\underline{b} = (b_1\ b_2 \ldots b_m)^T$ and a scalar x, we have

$$\underline{a}\circ\underline{b} = x \to (\underline{b}\alpha x)^T \geqslant \underline{a} \qquad (1)$$

$$\underline{a}\circ\underline{b} = x \to \exists\ \phi(\underline{b}\beta x) \varepsilon \Phi(\underline{b}\beta x) : (\phi(\underline{b}\beta x))^T \leqslant \underline{a} \qquad (2)$$

<u>Lemma 3</u> Given a mxn matrix $\underline{R} = [r_{ij}]$ and row vectors $\underline{a} = (a_1\ a_2 \ldots a_m)$, $\underline{b} = (b_1\ b_2 \ldots b_n)$, we have

$$\underline{a}\circ\underline{R} = \underline{b} \to \underline{a} \leqslant \bigwedge(\underline{R}\alpha\underline{b})^T \qquad (3)$$

$$\underline{a}\circ\underline{R} = \underline{b} \to \exists\ \phi(\underline{R}\beta\underline{b}) \varepsilon \Phi(\underline{R}\beta\underline{b}) : \bigvee(\phi(\underline{R}\beta\underline{b}))^T \leqslant \underline{a} \qquad (4)$$

<u>Theorem 1</u> Given a mxn matrix $\underline{R} = [r_{ij}]$ and a row vector $\underline{b} = (b_1\ b_2 \ldots b_n)$, we have

$$\exists \underline{a} : \underline{a}\circ\underline{R} = \underline{b} \Leftrightarrow \bigwedge(\underline{R}\alpha\underline{b})^T \circ \underline{R} = \underline{b} \qquad (5)$$

$$\exists \underline{a} : \underline{a}\circ\underline{R} = \underline{b} \Leftrightarrow \exists\ \phi(\underline{R}\beta\underline{b}) \varepsilon \Phi(\underline{R}\beta\underline{b}) : \bigvee(\phi(\underline{R}\beta\underline{b}))^T \circ \underline{R} = \underline{b} \qquad (6)$$

<u>Proof</u> (5): See Sanchez (1976)

(6): $\to)$ $\underline{a}\circ\underline{R} = \underline{b} \to \exists\ \phi(\underline{R}\beta\underline{b}) \varepsilon \Phi(\underline{R}\beta\underline{b}) : \bigvee(\phi(\underline{R}\beta\underline{b}))^T \leqslant \underline{a}$ (from Lemma 3)

$\to \bigvee(\phi(\underline{R}\beta\underline{b}))^T \circ \underline{R} \leqslant \underline{a}\circ\underline{R} = \underline{b}$. Let $\underline{r}_j$ be the jth column vector of $\underline{R}$ and $\phi(\underline{R}\beta\underline{b})$ $= [\phi_1(\underline{r}_1\beta b_1)\ \ \phi_2(\underline{r}_2\beta b_2) \ldots \phi_n(\underline{r}_n\beta b_n)]$.
Then $\bigvee(\phi(\underline{R}\beta\underline{b}))^T = (\phi_1(\underline{r}_1\beta b_1))^T \vee \ldots \vee(\phi_n(\underline{r}_n\beta b_n))^T \to \bigvee(\phi(\underline{R}\beta\underline{b}))^T \geqslant (\phi_j(\underline{r}_j\beta b_j))^T$, $\forall j \to \bigvee(\phi(\underline{R}\beta\underline{b}))^T \circ \underline{r}_j \geqslant (\phi_j(\underline{r}_j\beta b_j))^T \circ \underline{r}_j = b_j$, $\forall j$ (from Lemma 1) $\to \bigvee(\phi(\underline{R}\beta\underline{b}))^T \circ \underline{R} \geqslant \underline{b}$. Thus $\underline{b} \leqslant \bigvee(\phi(\underline{R}\beta\underline{b}))^T \circ \underline{R} \leqslant \underline{b}$, i.e., $\bigvee(\phi(\underline{R}\beta\underline{b}))^T \circ \underline{R} = \underline{b}$.

$\leftarrow)$ Obvious.

Theorem 1 states the necessary and sufficient conditions for the existence of a solution of the inverse problem. Thus, given a fuzzy relation R from $U = \{u_i/i=1,\ldots,m\}$ to $V = \{v_j/j=1,\ldots,n\}$

$$R = \{((u_i,v_j),r_{ij})/i=1,\ldots,m,j=1,\ldots,n\}$$

and a fuzzy subset B of V

$$B = \{(v_j,b_j)/\ j=1,\ldots,n\},$$

let $\underline{R} = [r_{ij}]$ be the mxn matrix corresponding to R and $\underline{b} = (b_1\ b_2\ldots b_n)$ the vector corresponding to B. Then the necessary and sufficient conditions for the existence of a fuzzy subset $A\subset U$ satisfying $A\circ R = B$, are given by either Eq. (5) or Eq. (6).

Obviously the two conditions are equivalent, implying each other, i.e.

$$\bigwedge(\underline{R}\alpha\underline{b})^T\circ\underline{R} = \underline{b} \Leftrightarrow \exists\phi(\underline{R}\beta\underline{b})\,\varepsilon\,\Phi(\underline{R}\beta\underline{b}) : \bigvee(\phi(\underline{R}\beta\underline{b}))^T\circ\underline{R} = \underline{b}.$$

4. SOLUTION OF THE INVERSE PROBLEM

4.1 δ-composition and its properties

Given a mxn matrix $\underline{R} = [r_{ij}]$ and a row vector $\underline{b} = (b_1\ b_2\ldots b_n)$, the δ-composition of $\underline{R}$ with $\underline{b}$, denoted by $\underline{R}\delta\underline{b}$, is defined by the mxn matrix

$$\underline{R}\delta\underline{b} \triangleq [s_{ij}],\ s_{ij} = (\bigwedge_{k=1}^{n}(r_{ik}\alpha b_k))\beta(r_{ij}\beta b_j),\ i=1,\ldots,m,\ j=1,\ldots,n$$

Note that $\bigwedge_{k=1}^{n}(r_{ik}\alpha b_k)$ is the ith element of the row vector $\bigwedge(\underline{R}\alpha\underline{b})^T$ and $r_{ij}\beta b_j$ is the (i,j)th element of the mxn matrix $\underline{R}\beta\underline{b}$. Thus $\underline{R}\delta\underline{b}$ can be obtained from $\bigwedge(\underline{R}\alpha\underline{b})^T$ and $\underline{R}\beta\underline{b}$.

Example: Let

$$\underline{R} = \begin{bmatrix} .7 & .5 & 1 & .5 \\ .5 & .2 & .7 & .6 \\ 1 & .8 & .3 & 0 \end{bmatrix}$$

$$\underline{b} = (.7 \quad .5 \quad .9 \quad .6)$$

We have

$$\underline{R}\alpha\underline{b} = \begin{bmatrix} 1 & 1 & .9 & 1 \\ 1 & 1 & 1 & 1 \\ .7 & .5 & 1 & 1 \end{bmatrix}$$

$$\bigwedge(\underline{R}\alpha\underline{b})^T = (.9 \quad 1 \quad .5)$$

$$\underline{R}\beta\underline{b} = \begin{bmatrix} .7 & .5 & .9 & 0 \\ 0 & 0 & 0 & .6 \\ .7 & .5 & 0 & 0 \end{bmatrix}$$

and finally

$$\underline{R}\delta\underline{b} = \begin{bmatrix} .7 & .5 & .9 & 0 \\ 0 & 0 & 0 & .6 \\ 0 & .5 & 0 & 0 \end{bmatrix}$$

Given $\underline{R}$, $\underline{b}$, we have

P9 $\underline{R}\delta\underline{b} \leqslant \underline{R}\beta\underline{b}$

P10 $\Phi(\underline{R}\delta\underline{b}) \dot{\subset} \Phi(\underline{R}\beta\underline{b})$

P11 $\bigvee(\phi(\underline{R}\delta\underline{b}))^T \leqslant \bigwedge(\underline{R}\alpha\underline{b})^T, \ \forall \phi(\underline{R}\delta\underline{b}) \in \Phi(\underline{R}\delta\underline{b})$

P12 $\bigvee(\phi(\underline{R}\beta\underline{b}))^T \leqslant \bigwedge(\underline{R}\alpha\underline{b})^T \Leftrightarrow \phi(\underline{R}\beta\underline{b}) \in \Phi(\underline{R}\delta\underline{b})$

4.2 The solution

Lemma 4 Given a column vector $\underline{b} = (b_1\ b_2 \ldots b_m)^T$ and a scalar x, assume that $\exists\ \underline{a}:\underline{a}\circ\underline{b} = x$. Then

$$\forall\ \underline{a}:\underline{a}\circ\underline{b} = x \Rightarrow \exists\phi(\underline{b}\beta x)\in\Phi(\underline{b}\beta x):(\phi(\underline{b}\beta x))^T \leq \underline{a} \leq (\underline{b}\alpha x)^T \quad (7)$$

$$\forall\underline{a},\ \forall\phi(\underline{b}\beta x)\in\Phi(\underline{b}\beta x):(\phi(\underline{b}\beta x))^T \leq \underline{a} \leq (\underline{b}\alpha x)^T \Rightarrow \underline{a}\circ\underline{b} = x. \quad (8)$$

Theorem 2 Given a mxn matrix $\underline{R}$ and a row vector $\underline{b} = (b_1\ b_2 \ldots b_n)$, assume that $\exists\ \underline{a}:\underline{a}\circ\underline{R} = \underline{b}$. Then

$$\forall\ \underline{a}:\underline{a}\circ\underline{R} = \underline{b} \Rightarrow \exists\phi(\underline{R}\delta\underline{b})\in\Phi(\underline{R}\delta\underline{b}): \bigvee(\phi(\underline{R}\delta\underline{b}))^T \leq \underline{a} \leq \bigwedge(\underline{R}\alpha\underline{b})^T \quad (9)$$

$$\forall\ \underline{a},\forall\phi(\underline{R}\delta\underline{b})\in\Phi(\underline{R}\delta\underline{b}): \bigvee(\phi(\underline{R}\delta\underline{b}))^T \leq \underline{a} \leq \bigwedge(\underline{R}\alpha\underline{b})^T \Rightarrow \underline{a}\circ\underline{R} = \underline{b} \quad (10)$$

Proof (9): $\underline{a}\circ\underline{R} = \underline{b} \rightarrow \exists\phi(\underline{R}\beta\underline{b})\in\Phi(\underline{R}\beta\underline{b}): \bigvee(\phi(\underline{R}\beta\underline{b}))^T \leq \underline{a} \leq \bigwedge(\underline{R}\alpha\underline{b})^T$ (from Lemma 3) $\rightarrow$
$\exists\phi(\underline{R}\delta\underline{b})\in\Phi(\underline{R}\delta\underline{b}): \bigvee(\phi(\underline{R}\delta\underline{b}))^T \leq \underline{a} \leq \bigwedge(\underline{R}\alpha\underline{b})^T$ (from P12).
(10): $\bigvee(\phi(\underline{R}\delta\underline{b}))^T \leq \underline{a} \leq \bigwedge(\underline{R}\alpha\underline{b})^T \rightarrow \bigvee(\phi(\underline{R}\beta\underline{b}))^T \leq \underline{a} \leq \bigwedge(\underline{R}\alpha\underline{b})^T$, $\phi(\underline{R}\beta\underline{b})\in\Phi(\underline{R}\beta\underline{b})$ (from P10) $\rightarrow (\phi_j(\underline{r}_j\beta b_j))^T \leq \underline{a} \leq (\underline{r}_j\alpha b_j)^T, \forall j$, where $\underline{r}_j$ is the jth column vector of $\underline{R}$, $\rightarrow \underline{a}\circ\underline{r}_j = b_j, \forall j$ (from Lemma 4) $\rightarrow \underline{a}\circ\underline{R} = \underline{b}$.

The solution of the inverse problem is derived from theorem 2 as follows:

Given the fuzzy relation R and the fuzzy subset B, all fuzzy subsets A such that A∘R = B are given by

$$\bigvee(\phi(\underline{R}\delta\underline{b}))^T \leq \underline{a} \leq \bigwedge(\underline{R}\alpha\underline{b})^T,\ \forall\phi(\underline{R}\delta\underline{b})\in\Phi(\underline{R}\delta\underline{b})$$

provided that there exists at least one such A, where

$\underline{R}$: the matrix corresponding to R

$\underline{a},\underline{b}$: the vectors corresponding to A, B respectively.

4.3 Example: Let

R =

U \ V	v_1	v_2	v_3	v_4	v_5
u_1	.4	0	.9	.6	.8
u_2	.7	.8	.3	1.	.5
u_3	.6	.4	.3	.4	.9
u_4	.2	1.	.5	.8	.4

B=

V	v_1	v_2	v_3	v_4	v_5
$\underline{b}$	.6	.5	.9	.6	.8

We have

$$\underline{R}\alpha\underline{b} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ .6 & .5 & 1 & .6 & 1 \\ 1 & 1 & 1 & 1 & .8 \\ 1 & .5 & 1 & .6 & 1 \end{pmatrix}$$

$$\bigwedge(\underline{R}\alpha\underline{b})^T = (1 \quad .5 \quad .8 \quad .5)$$

$$\underline{R}\beta\underline{b} = \begin{pmatrix} 0 & 0 & .9 & .6 & .8 \\ .6 & .5 & 0 & .6 & 0 \\ .6 & 0 & 0 & 0 & .8 \\ 0 & .5 & 0 & .6 & 0 \end{pmatrix}$$

$$\underline{R}\delta\underline{b} = \begin{pmatrix} 0 & 0 & .9 & .6 & .8 \\ 0 & .5 & 0 & 0 & 0 \\ .6 & 0 & 0 & 0 & .8 \\ 0 & .5 & 0 & 0 & 0 \end{pmatrix}$$

It is easily seen that $\bigwedge(\underline{R}\alpha\underline{b})^{T}\circ\underline{R} = \underline{b}$, thus $\exists\,\underline{a}:\underline{a}\circ\underline{R} = \underline{b}$. Consider, for example, $\phi_1(\underline{R}\delta\underline{b})\varepsilon\Phi(\underline{R}\delta\underline{b})$ such that

$$\phi_1(\underline{R}\delta\underline{b}) = \begin{pmatrix} 0 & 0 & .9 & .6 & 0 \\ 0 & .5 & 0 & 0 & 0 \\ .6 & 0 & 0 & 0 & .8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have

$$V(\phi_1(\underline{R}\delta\underline{b}))^{T} = (.9 \quad .5 \quad .8 \quad 0).$$

Any fuzzy subset A, with grades of membership vector $\underline{a}$ such that

$$(.9 \quad .5 \quad .8 \quad 0) \leqslant \underline{a} \leqslant (1 \quad .5 \quad .8 \quad .5)$$

has the property that $A\circ R = B$. Now consider $\phi_2(\underline{R}\delta\underline{b})\varepsilon\Phi(\underline{R}\delta\underline{b})$, such that

$$\phi_2(\underline{R}\delta\underline{b}) = \begin{pmatrix} 0 & 0 & .9 & .6 & .8 \\ 0 & .5 & 0 & 0 & 0 \\ .6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have

$$V(\phi_2(\underline{R}\delta\underline{b}))^{T} = (.9 \quad .5 \quad .6 \quad 0)$$

Again, any fuzzy subset A, with grades of membership vector $\underline{a}$ such that

$$(.9 \quad .5 \quad .6 \quad 0) \leq \underline{a} \leq (1 \quad .5 \quad .8 \quad .5),$$

has the property that A∘R = B. Note, however, that

$$(.9 \quad .5 \quad .6 \quad 0) \leq (.9 \quad .5 \quad .8 \quad 0)$$

4.4 Remark

$\bigwedge(\underline{R}\alpha\underline{b})^T$ is the least upper bound (l.u.b.) of the solution vectors of the inverse problem. The set $\{\bigvee(\phi(\underline{R}\delta\underline{b}))^T / \phi(\underline{R}\delta\underline{b}) \in \Phi(\underline{R}\delta\underline{b})\}$ includes the lower bounds of the solution vectors. Generally, we may have the case, where

$$\bigvee(\phi_1(\underline{R}\delta\underline{b}))^T < \bigvee(\phi_2(\underline{R}\delta\underline{b}))^T$$

as it has been seen in the last example. In the next section, a class of non-greatest lower bounds is identified, the solution of the inverse problem being thus improved.

5. IMPROVEMENT OF THE SOLUTION

5.1 Definitions

Let a mxn matrix $\underline{R} = [r_{ij}]$ and a row vector $\underline{b} = (b_1\ b_2 \ldots b_n)$. In the sequel, the set $\{\bigvee(\phi(\underline{R}\delta\underline{b}))^T / \phi(\underline{R}\delta\underline{b}) \in \Phi(\underline{R}\delta\underline{b})\}$ will be denoted by $\bigvee(\Phi(\underline{R}\delta\underline{b}))^T$.

A vector $\bigvee(\phi_1(\underline{R}\delta\underline{b}))^T \in \bigvee(\Phi(\underline{R}\delta\underline{b}))^T$ is said to be redundant if there exists $\bigvee(\phi_2(\underline{R}\delta\underline{b}))^T \in \bigvee(\Phi(\underline{R}\delta\underline{b}))^T$ such that

$$\bigvee(\phi_2(\underline{R}\delta\underline{b}))^T \leq \bigvee(\phi_1(\underline{R}\delta\underline{b}))^T$$

Let $\underline{s}_k$, $\underline{s}_\ell$ be the kth, ℓth column vectors of matrix $\underline{S}$ respectively. If

$s_{i\ell} \neq 0 \rightarrow s_{ik} \neq 0$ and $s_{ik} \leqslant s_{i\ell}, \forall i$

$(s_{i\ell} = 0 \rightarrow s_{ik}$ is arbitrary)

then $\underline{s}_k$ is said to be dominated by $\underline{s}_\ell$.

5.2 The improvement of the solution of the inverse problem

Let a mxn matrix $\underline{R} = [r_{ij}]$ and a row vector $\underline{b} = (b_1\ b_2 \ldots b_n)$ and assume that $\exists \underline{a}: \underline{a} \circ \underline{R} = \underline{b}$. Let $\underline{S} = \underline{R} \delta \underline{b} = [s_{ij}]$. If $\underline{S}_0$ is the matrix obtained from $\underline{S}$ by deleting all its zero column vectors, it can be shown that

$$\bigvee(\Phi(\underline{S}))^T = \bigvee(\Phi(\underline{S}_0))^T \tag{11}$$

Denoting by $\underline{S}_k$ the matrix obtained from $\underline{S}_0$ by deleting a column vector $\underline{s}_k$, it can also be shown that if and only if $\underline{s}_k$ is dominated, then

$$\bigvee(\Phi(\underline{S}_k))^T \subset \bigvee(\Phi(\underline{S}))^T \tag{12}$$

and

$$\exists\, \phi(\underline{S}_k) \varepsilon \Phi(\underline{S}_k): \bigvee(\phi(\underline{S}))^T \geqslant \bigvee(\phi(\underline{S}_k))^T, \forall \phi(\underline{S}) \varepsilon \Phi(\underline{S}) \tag{13}$$

The significance of (12) and (13) is that, by deleting a dominated vector $\underline{s}_k$ from $\underline{S}_0$, a (possibly empty) class of redundant vectors is excluded from the solutions which are obtained from

$$\bigvee(\phi(\underline{S}_k))^T \leqslant \underline{a} \leqslant \bigwedge(\underline{R} \alpha \underline{b})^T, \forall \phi(\underline{S}_k) \varepsilon \Phi(\underline{S}_k).$$

Thus, if z, z' are the number of row vectors in $\bigvee(\Phi(\underline{S}))^T$ and $\bigvee(\Phi(\underline{S}_k))^T$ respectively, we have

$$z' = \frac{z}{z_k}$$

where z_k is the number of nonzero elements in the column vector $\underline{s}_k$.

If $\underline{S}^*$ is the matrix obtained from $\underline{S}_0$ by deleting all its dominated column vectors, we further obtain from (12) and (13) that

$$\bigvee(\Phi(\underline{S}^*))^T \subset \bigvee(\Phi(\underline{S}))^T \quad (14)$$

and

$$\exists\, \phi(\underline{S}^*) \varepsilon \Phi(\underline{S}^*) : \bigvee(\phi(\underline{S}))^T \geq \bigvee(\phi(\underline{S}^*))^T, \forall \phi(\underline{S}) \varepsilon \Phi(\underline{S}) \quad (15)$$

Thus all the solutions of the inverse problem are given by

$$\bigvee(\phi(\underline{S}^*))^T \leq \underline{a} \leq \bigwedge(\underline{R}\alpha\underline{b})^T, \forall \phi(\underline{S}^*) \varepsilon \Phi(\underline{S}^*).$$

It can be shown that any reduction of the dimensions of $\underline{S}^*$ would result in some nonredundant vectors of $\bigvee(\Phi(\underline{S}))^T$ being eliminated. However, some redundant vectors may still be included in $\bigvee(\Phi(\underline{S}^*))^T$.

5.3 Example

In the last example (section 4) we had

$$\underline{S} = \underline{R}\delta\underline{b} = \begin{pmatrix} 0 & 0 & .9 & .6 & .8 \\ 0 & .5 & 0 & 0 & 0 \\ .6 & 0 & 0 & 0 & .8 \\ 0 & .5 & 0 & 0 & 0 \end{pmatrix}$$

and $\bigwedge(\underline{R}\alpha\underline{b})^T = (1 \quad .5 \quad .8 \quad .5)$.

Thus

$$\underline{S}^* = \begin{pmatrix} 0 & 0 & .9 \\ 0 & .5 & 0 \\ .6 & 0 & 0 \\ 0 & .5 & 0 \end{pmatrix}$$

since the 4th and 5th column vectors of $\underline{S}$ are dominated by the 3rd column vector. We have

$$\bigvee(\Phi(\underline{S}^*))^T = \{(.9 \quad .5 \quad .6 \quad 0), (.9 \quad 0 \quad .6 \quad .5)\}$$

All $\underline{a}$'s, such that $\underline{a} \circ \underline{R} = \underline{b}$ are given by

$$(.9 \quad .5 \quad .6 \quad 0) \leqslant \underline{a} \leqslant (1 \quad .5 \quad .8 \quad .5)$$

or

$$(.9 \quad 0 \quad .6 \quad .5) \leqslant \underline{a} \leqslant (1 \quad .5 \quad .8 \quad .5)$$

These are shown by the tree of Fig. 1.

6. CONCLUSIONS

In this paper, the inverse problem of fuzzy relational equations has been investigated. A different type of the conditions for the existence of a solution have been derived, and the problem is given an analytical solution. A method has been proposed, by means of which a class of redundant lower bounds of the solutions are readily eliminated; the general solution is thus improved.

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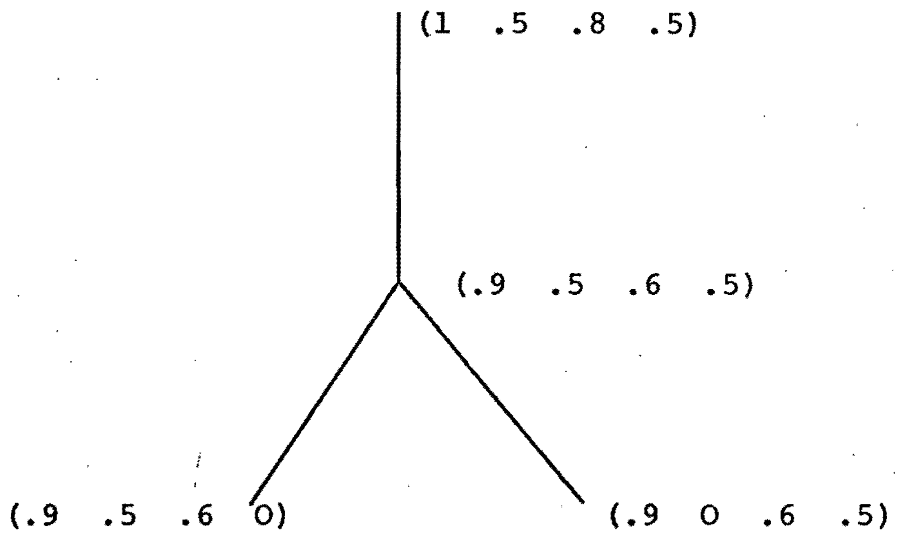

Figure 1. Solution tree

ARRANGEMENT OF FORMULAS AND MINIMIZATION IN Pi-LOGICS (ALGEBRAS).

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Summary.

It is shown that all canonic formulas in Pi-logics are generally minimizeable, this possibility depending again only on the defined or crucial properties of the connectives. With some further restrictions imposed on the connectives other types of canonic formulas can be constructed as well, which are again minimizeable.

Introduction.

The paper presupposes knowledge of (1) and preferrably of (2),(3) and (4).

It was shown in (1) and quoted in (2) and (3), that three incompletely defined connectives of two arguments, i.e.

a) $v_1 \boxplus v_2 = \begin{cases} 0 \text{ if } v_i = 0 \\ v_i \text{ if } v_j = 1 \end{cases}$

b) $v_1 \oplus v_2 = \begin{cases} 0 \text{ iff } v_i = 0 \\ 1 \text{ if } v_i = 1 \end{cases}$ otherwise undefined

c) $v_1 \odot v_2 = v_i \text{ if } v_j = 0$

plus the cyclic negation defined as: $v+1 \pmod{k} = \tilde{v}$

always form a functionally complete system in any finite multiple-valued logic, as every function $f(v_1, v_2, \dots, v_n) \neq 0$ can always be expressed by the canonic type of formula:

$$\bigodot_{\substack{\lambda=1 \\ f(a_1,a_2,\dots,a_n)\neq 0}}^{m \leq k^n} \left[c_\gamma \boxplus \left(\bigoplus_{l=1}^{n} \psi_{a_l}(v_l) \right) \right]_\lambda$$

The constants and the characteristic functions of the type: $\psi_i(v) = \begin{cases} 1 \text{ if } v = i \\ 0 \text{ otherwise} \end{cases}$ can be generated by the subset $\{\oplus, \sim\}$ as shown in the papers quoted.

Thus the method enables convenient generating of new types of connectives which will always form a functionally complete system, provided their crucial or defining properties have been preserved.

Further work in this topic was concerned with additional properties of the connectives which would enable forming complete systems consisting of less than four connectives.

It was also shown that most, if not all, of the multiple-valued logics used in practice are actually instances of the Pi-logics or show an obvious and close relationship with them.

However, the canonic formulas are clumsy and very legthy. Thus, it was obvious that, should the Pi-logics have any practical applicability, the question of minimization of the canonic formulas would require some attention. It seemed that - owing to the incomplete definitions of the general types of connectives - the question of minimizeability would have to be considered separately in every instance. Rather surprisingly, the opposite was found to apply. This is the subject of this paper.

Whilst the basic facts have been repeated here, a deeper understanding of the topic would require familiarization with the previous papers quoted.

Minimization in Pi-logics (algebras).

As shown elsewhere /(1),(2),(3),(4)/, the following holds for any Pi-algebra:

$$\underset{\substack{\lambda=1\\ f(a_1,a_2,\dots,a_n)\neq 0}}{\overset{m\leq k^n}{\odot}} \left[c_\gamma \boxplus \left(\underset{l=1}{\overset{n}{\Diamond}} \psi_{a_l}(v_l) \right) \right]_\lambda = \underset{\substack{\lambda=1\\ f(a_1,a_2,\dots,a_n)\neq 0}}{\overset{m\leq k^n}{\odot}} \left[c_\gamma \boxplus \left(\underset{l=1}{\overset{n}{\boxplus}} \psi_{a_l}(v_l) \right) \right]_\lambda ,$$

this owing to the fact that the functions of the type: $\psi_i(v)$ can assume only the values 0,1 and for these values the connectives $\boxplus$ and $\Diamond$ behave the sam *-s apparent from their respective definitions.

Let us now consider a subformula of the type:

$$\left(\underset{l=1}{\overset{n}{\boxplus}} \psi_{a_l}(v_l) \right)_\lambda$$

Taking any two of the expressions of this type, it is clear that they cannot both assume the value of 1 for the same substitution. It is, however, possible to have such a substitution where both expressions of the given type assume the value of 0. / It should be realized that, whilst defined for the pairs of values $\langle 00 \rangle$, $\langle 01 \rangle$, $\langle 10 \rangle$, the general connective '$\odot$' is not defined for the pair of values $\langle 1,1 \rangle$ ./

Let us denote the above type of expressions with φ_1, φ_2 . Further, let a be any elementary formula / partial expression / such as may occur in a canonic formula of the type considered. Bearing in mind that φ_1, φ_2 cannot assume both the value of 1 at the same time, it is easy to see that the following holds for the possible substitutions:

$$(a \boxplus \varphi_1) \odot (a \boxplus \varphi_2) = a \boxplus (\varphi_1 \odot \varphi_2) ,$$

where both sides of the equation assume the value of 0 for the substitution $\langle 00 \rangle$ and the value a for the remaining possible substitutions $\langle 10 \rangle$ and $\langle 01 \rangle$.

Let us further realize that: $\bigodot_{i=0}^{k-1} \psi_i(v) = 1$, as is obvious.
It is then possible to formulate Theorem 1.

Theorem 1.

In canonic expressions of the type:

$$f(v_1, v_2, \dots, v_n) = \bigodot_{\substack{\lambda=1 \\ f(a_1,a_2,\dots a_n)\neq 0}}^{m\leq k^n} \left[c_\gamma \boxplus \left(\boxplus_{l=1}^{n} \psi_{a_l}(v_l) \right) \right]_\lambda$$

any connectives of the type $\boxplus$, $\odot$ will behave in such a way so that $\boxplus$ will appear distributive with respect to $\odot$,/ although not necessarily so in a general context /.
Also, owing to the fact that $\bigodot_{i=0}^{k-1} \psi_i(v) = 1$, the above type of formula will be generally minimizeable.

The Proof preceded the Theorem.

This gives a basic approach to minimization. Clearly, minimization algorithms or strategies developed for canonic formulas in modular logics may be applied straight to canonic formulas in any Pi-logic emploing the $\boxplus$ -type. This follows from the fact that the modular operations:"summation and multiplication modulo k" are instances of $\odot$ and $\boxplus$ respectively and the other properties of individual instances of $\odot$, $\boxplus$ do not influence the process of minimization, as already shown.

Let us now consider the type of function: $\diamondsuit \subset \{\diamondsuit\}$ having the additional property of: $v_i \diamondsuit k-1 = v_i$.
As shown elswhere (4) *) the set $\{\diamondsuit, \sim\}$ forms a functionally complete system.

Let us further introduce the type of function:

$$f_i(v) = \begin{cases} k-1 & \text{iff } v=i \\ 0 & \text{otherwise} \end{cases}$$

As shown again in (4), these functions can be generated out of the set $\{\diamondsuit, \sim\}$ by first generating the functions $\psi_i(v)$ in the familiar way / see (1),(2),(3)/ and then applying the formula:

$$f_i(v) = \diamondsuit \left[\{ \overset{\sim k-1}{\psi_x(v)} \} - \overset{\sim k-1}{\psi_i(v)} \right]$$

The constants C_γ are of course also generable via $\{\diamondsuit, \sim\}$.

Let us further introduce any function of the type $\odot$. The set $\{\diamondsuit, \sim\}$ allows two specific subtypes of $\odot$ to be generated, but this is irrelevant from the point of view of the problem now considered.

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*)The respective Theorem has been added to the final version of (3). However, as stated on pg 475 of the respective Proceedings the final version had not arrived in time, whereupon the organizers kindly had the original draft re-typed and published. /author's note/.

Lemma 1.

For any $f(v_1,v_2,\dots,v_n) \neq 0$ the following holds:

$$f(v_1,v_2,\dots,v_n) = \underset{f(a_1,a_2,\dots,a_n)\neq 0}{\overset{m\leq k^n}{\underset{\lambda=1}{\bigodot}}} [c_\gamma \diamond (\overset{n}{\underset{l=1}{\Diamond}} f_{a_l}(v_l))]_\lambda$$

Proof.

The correctness of this Lemma is obvious. If a value vector $\langle a_1,a_2,\dots,a_n\rangle$ fits the expression$(\overset{n}{\underset{l=1}{\Diamond}} f_{a_l}(v_l))_\lambda$ then it does not fit any other expression of this type.

Consequently, for that particular expression it is: $\overset{n}{\underset{l=1}{\Diamond}} f_{a_l}(v_l) = k-1$ owing to the properties of $\diamond$. Then $c_\gamma \diamond k-1 = c_\gamma$ and bearing in mind that all the remaining expressions$[c_\gamma \diamond (\overset{}{\underset{l=1}{\Diamond}} f_{a_l}(v_l))]\lambda = 0$, the formula will be reduced to $c_\gamma \odot 0$ or $0 \odot c_\gamma = 0$. Similarly for all the other value vectors. Thus the above holds. Q.E.D.

Let us now consider any two expressions of the type $[\overset{n}{\underset{l=1}{\Diamond}} f_{a_l}(v_l)]_\lambda$, both pertaining to an instance of the general type of formula Lemma 1.

Let us symbolize them by ρ_1, ρ_2. Obviously, the expressions ρ_1, ρ_2 cannot both assume the value $k-1$ for the same substitution etc. as with Proof of Theorem 1.

It is then easy to see that the following holds for the possible pairs of value substitutions: $\langle 0,0\rangle, \langle k-1,0\rangle, \langle 0,k-1\rangle$:

$$a \diamond (\rho_1 \odot \rho_2) = (a \diamond \rho_1) \odot (a \diamond \rho_2)$$

Let us further realize that:

$\overset{k-1}{\underset{i=0}{\bigodot}} f_i(v) = k-1$, [so that also : $\overset{k-1}{\underset{i=0}{\textcircled{V}}} f_i(v) = k-1$, (where $\textcircled{V} \in \{\oplus\}$ and $v_i \textcircled{V} k-1 = k-1$)]

It is then easy to see the correctness of the following Theorem 1 a.

Theorem 1 a .

In canonic expressions of the type:

$$f(v_1,v_2,\dots,v_n) = \underset{f(a_1,a_2,\dots,a_n)\neq 0}{\overset{m\leq k^n}{\underset{\lambda=1}{\bigodot}}} [c_\gamma \diamond (\overset{n}{\underset{l=1}{\Diamond}} f_{a_l}(v_l))]_\lambda$$

any pair of connectives of the type: $\odot, \diamond$ will behave in such a way so that $\diamond$ will appear to be distributive with respect to $\odot$ in that type of formula, although not necessarily so in general. Owing further to: $\overset{k-1}{\underset{i=0}{\bigodot}} f_i(v) = k-1$, the above type of formula will be generally minimizeable.

The Proof preceded the Theorem.

Let us now consider the following functions:

$$\boxminus \subset \{\boxplus\}; \quad v \boxminus v = v$$

$$\ominus \subset \{\odot\}; \quad v \ominus v = v$$

Let us further introduce the following characteristic functions:

$$f_i^{\varkappa}(v) = \begin{cases} \varkappa \text{ iff } v = i \\ 0 \text{ otherwise} \end{cases}$$

where: $\varkappa = 1, 2, \ldots, k-1$; $i = 0, 1, 2, \ldots, k-1$

If any system of the type: $\{\boxminus, \ominus\}$ is completed by any system of the type $\{\oplus, \sim\}$, then the above functions can be generated via the subset $\{\oplus, \sim\}$. Constants and characteristic functions $\psi_i(v)$ may be of course generated by $\{\oplus, \sim\}$ and: $\varkappa \boxminus \psi_i(v) = f_i^{\varkappa}(v)$.

We can now formulate

<u>Lemma 2</u>.

For any multiple-valued function $f(v_1, v_2, \ldots, v_n) \neq 0$ the following holds:

$$f(v_1, v_2, \ldots, v_n) = \underset{\substack{\lambda=1 \\ f(a_1, a_2, \ldots a_n) \neq 0}}{\overset{m \leq k^n}{\ominus}} \Big[\underset{l=1}{\overset{n}{\boxminus}} f_{a_l}^{\varkappa}(v_l) \Big]_\lambda$$

<u>Proof</u>.

Considering the basic canonic formula in Pi-logics / such as shown, e.g. here in Theorem 1,/ we realize that the expressions of the type $[C_\gamma \boxplus |_{l=1}^{n} \boxminus \psi_{a_l}(v_l))]_\lambda$ can be replaced by those of the type $[\underset{l=1}{\overset{n}{\boxminus}} f_{a_l}^{\varkappa}(v_l)]_\lambda$, as they will obviously behave the same way with respect to the value vectors. Thus, the above holds. Q.E.D.

/ In this full canonic form, i.e. not yet considering minimization, the '$\odot$'-type need not be of the subclass '$\ominus$'./

We may further observe that, with respect to the characteristic functions of the type: $f_i^{\varkappa}(v)$ / with the same $\varkappa$ /, the connectives will behave in a wa isomorphic to the "&" and "V" in binary logic. Clearly, as the functions $f_i^{\varkappa}(v)$ may assume only the values $\varkappa$, 0, we have:

$\boxminus$	$\varkappa$	0
$\varkappa$	$\varkappa$	0
0	0	0

$\ominus$	$\varkappa$	0
$\varkappa$	$\varkappa$	$\varkappa$
0	$\varkappa$	0

The following then holds / and may be verified by the table method /:

a) $f_i^{\varkappa}(v_1) \boxminus (f_j^{\varkappa}(v_2) \ominus f_l^{\varkappa}(v_3)) = (f_i^{\varkappa}(v_1) \boxminus f_j^{\varkappa}(v_2)) \ominus (f_i^{\varkappa}(v_1) \boxminus f_l^{\varkappa}(v_3))$

b) $f_i^{\varkappa}(v_1) \ominus (f_j^{\varkappa}(v_2) \boxminus f_l^{\varkappa}(v_3)) = (f_i^{\varkappa}(v_1) \ominus f_j^{\varkappa}(v_2)) \boxminus (f_i^{\varkappa}(v_1) \ominus f_l^{\varkappa}(v_3))$

Let us further realize, that $\bigominus_{i=0}^{k-1} f_i^{\varkappa}(v) = \varkappa$

and: $f_i^{\varkappa}(v) \boxminus \varkappa = f_i^{\varkappa}(v)$; (same $\varkappa$).

It is also: $f_i^{\varkappa}(v) \boxminus f_j^{\varkappa}(v) = 0$.

Thus, if rendered by means of $\ominus$, $\boxminus$ and $f_i^{\varkappa}(v_l)$, any multiple-valued function may be viewed as consisting of maximally $k-1$ "pseudobinary" functions, which can be each arranged like real binary functions and then minimized following the rules set above.
We can thus formulate:

Theorem 2.

A formula given in the canonic form:

$$f(v_1, v_2, \dots, v_n) = \bigominus_{\substack{\lambda=1 \\ f(a_1, a_2, \dots, a_n) \neq 0}}^{m \leq k^n} \Big[\bigboxminus_{l=1}^{n} f_{a_l}^{\varkappa}(v_l) \Big]_{\lambda}$$

can generally be minimized by decomposition into $k-1$ or less pseudobinary functions in which case the connectives $\ominus$, $\boxminus$ are mutually distributive, so that the respective subformulae may be conveniently arranged and then reduced owing to the rules: $f_i^{\varkappa}(v) \boxminus f_j^{\varkappa}(v) = 0$,

$$\bigominus_{i=0}^{k-1} f_i^{\varkappa}(v) = \varkappa,$$

$$f_i^{\varkappa}(v) \boxminus \varkappa = f_i^{\varkappa}(v).$$

The Proof of the Theorem was given in the preceding demonstration.

It should be acknowledged that this approach using the functions of the type $f_i^{\varkappa}(v)$ and the idempotent subclasses of $\odot$, $\boxplus$ depends on a certain generalization of the ideas contained in a paper by Rabinovich and Ivas'kiv, dealing with the ternary case. (5). The generalization consists of an extension from k=3 to a general k and replacing the fully defined connectives by classes with just the necessary relevant properties. These, in their own turn form subclasses of still more general classes of Pi- connectives.

The canonic formulas dealt with so far may be considered as generalized analogies of complete disjunctive normal forms in binary logic.

Let us now consider a subclass of connectives such that: $\oplus \subset \{\odot\}$ and $v_i \oplus 1 = 1$. Let us further introduce the following type of characteristic function:

$$\tilde{f}_i^1(v) = \begin{cases} 0 & \text{iff } v = i \\ 1 & \text{otherwise} \end{cases}$$

/ Having any complete Pi-set, the $f_i(v)$ fuctions are of course generable out of $\{\boxplus, \oplus, \sim\}$ via the formula: $k-1 \boxplus \varphi_i(v)$ /.
It is therefore easy to see that the following Theorem 3 holds.

Theorem 3.

For any multiple-valued logical function $f(v_1, v_2, \ldots, v_n) \neq 0$, it holds that:

$$f(v_1, v_2, \ldots, v_n) = \bigodot_{\substack{\lambda=1 \\ f(a_1, a_2, \ldots, a_n) \neq 0}}^{\lambda = m \leq k^n} \Big[c_\gamma \boxplus \Big(\boxplus_{l=1}^{n} \varphi_{a_l}(v_l) \Big) \Big]_\lambda = \boxplus_{\substack{\lambda'=1 \\ f(a_1, a_2, \ldots, a_n) \neq 1}}^{m' = k^n - m} \Big[c_\varkappa \oplus \Big(\bigoplus_{l=1}^{n} \tilde{f}_{a_l}^1(v_l) \Big) \Big]_{\lambda'};$$

$$(c_\varkappa = 0, 2, \ldots\ldots, k-1)$$

Proof.

Any expression of the type $\Big(\bigoplus_{l=1}^{n} \tilde{f}_{a_l}^1(v_l) \Big)_{\lambda'} \oplus c_{\varkappa}$ will assume the value of $c_\varkappa$ if substituted by its respective value vector $\langle a_1, a_2, \ldots, a_n \rangle$ or else the value of 1, owing to the additional property of the $\oplus$ -type, i.e.: $v \oplus 1 = 1$. Thus, if the m' expressions of the type: $\Big[\bigoplus_{l=1}^{n} \tilde{f}_{a_l}^1(v_l) \Big]_{\lambda'}$ are formed, using characteristic functions pertaining to those value vectors for which the function does not assume the value of 1, it is clear that, if these values are fed into such a formula, the expression will assume the value of the respective constant $c_\varkappa \neq 1$.
(As $\bigoplus \tilde{f}_{a_l}^1(v_l) = 0$ when the correct values are substituted, it is not necessary to use an explicit constant 0).
On the other hand, if values which do not fit any expression of the type $\Big[\bigoplus \tilde{f}_{a_l}^1(v_l) \Big]_{\lambda'}$ are substituted into the formula, then clearly the whole formula will assume the value of 1. However, such substitutions are exactly those which have been left out when constructing the formula , and these are those, for which the function $f(v_1, v_2, \ldots, v_n) = 1$. The Theorem is proved.

Owing to the additional property of the $\oplus$ -subtype, i.e. $v_i \oplus 1 = 1$, it is also true that: $1 \oplus 1 = 1$. It is then easy to see that, with regard to the characteristic functions $\check{f}_i^1(v)$ the connectives $\oplus$, $\boxplus$ will behave as mutually distributive.

It is further obvious that in the type of formula Theorem 3, two expressions of the type $\left[\bigoplus_{\ell=1} \tilde{f}^{1}_{a_\ell}(v_\ell)\right]_{\lambda'}$ cannot assume the value of 0 for the same substitution. Denoting any two such expressions by: F_1, F_2 we may see that the following equation holds for the substitutions $\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,1\rangle$:

$$a \oplus (F_1 \boxplus F_2) = (a \oplus F_1) \boxplus (a \oplus F_2)$$

(Note: If the substitution $\langle 0,0\rangle$ could occur, a general '$\boxplus$'-type could not be used in the above formula, since, generally, '$\boxplus$' is not defined for $a \boxplus a$. However, such substitution cannot occur, as has been shown.)

Realizing further that $\boxplus_{i=1}^{k-1} \tilde{f}^{1}_{i}(v) = 0$ and $a \oplus 0 = a$ we may formulate the following Theorem 4.

Theorem 4.

In a canonic form of the type:

$$\boxplus_{\substack{\lambda'=1 \\ f(a_1,a_2,\dots,a_n)\neq 1}}^{m'=k^n-m} \left[c_\varkappa \oplus \left(\bigoplus_{\ell=1}^{n} \tilde{f}^{1}_{a_\ell}(v_\ell)\right)\right]_{\lambda'}$$

any connectives of the defined types will behave in such a way that $\oplus$ will appear as distributive with respect to $\boxplus$.

Owing further to $\boxplus_{i=1}^{k-1} \tilde{f}^{1}_{i}(v) = 0$ and $a \oplus 0 = a$, the above formula will be generally minimizeable.

The Proof preceded the Theorem.

Let us further consider again the function: $\diamondsuit \subset \{\oplus\}$ and $v_i \diamondsuit k-1 = v_i$. Let us introduce also the function $\textcircled{V} \subset \{\odot\}$ and $v_i \textcircled{V} k-1 = k-1$, and the function:

$$\tilde{\psi}^{k-1}_{i}(v) = \begin{cases} 0 \text{ iff } v = i \\ k-1 \text{ otherwise} \end{cases}$$

It is thus possible to formulate two Theorems:

Theorem 4a.

For any multiple-valued logical function $f(v_1, v_2, \dots, v_n) \neq 0$:

$$f(v_1,v_2,\dots v_n) = \bigodot_{\substack{\lambda=1 \\ f(a_1,a_2,\dots,a_n)\neq 0}}^{m} \left[c_\gamma \diamondsuit \left(\diamondsuit_{\ell=1}^{n} f_{a_\ell}(v_\ell)\right)\right]_\lambda = \diamondsuit_{\substack{\lambda'=1 \\ f(a_1,a_2,\dots,a_n)\neq k-1}}^{m'=k^n-m} \left[c_\varkappa \textcircled{V} \left(\textcircled{V}_{\ell=1}^{n} \tilde{\psi}^{k-1}_{a_\ell}(v_\ell)\right)\right]_{\lambda'}, \quad (c_\varkappa = 0,1,2,\dots k-2)$$

Theorem 4b.

In a canonic formula of the type:

$$\diamondsuit_{\substack{\lambda'=1 \\ f(a_1,a_2,\dots,a_n)\neq k-1}}^{m'=k^n} \left[c_\varkappa \textcircled{V} \left(\textcircled{V}_{\ell=1}^{n} \tilde{\psi}^{k-1}_{a_\ell}(v_\ell)\right)\right]_{\lambda'}$$

any connectives having the general properties defining $\textcircled{V}$, $\diamondsuit$ will behave in such a way so that $\textcircled{V}$ will appear as distributive with respect to $\diamondsuit$.

Owing further to $\overset{k-1}{\underset{i=0}{\Diamond}} \overset{\sim k-1}{\psi_i}(v) = 0$,and $a \textcircled{V} 0 = a$, this type of formula will be generally minimizeable.

Proofs of the Theorems 4a, 4b parallel exactly those of the Theorems 3, 4 and may be easily made explicit by substituting in the preceding Proofs in the following way:

$$\oplus \leftrightarrow \textcircled{V}, \quad \boxplus \leftrightarrow \Diamond, \quad 1 \leftrightarrow k-1, \quad \overset{\sim 1}{f_i}(v) \leftrightarrow \overset{\sim k-1}{\psi_i}(v), \quad \odot \leftrightarrow \odot, \quad 0 \leftrightarrow 0.$$

It may be observed that the formulas employing $\textcircled{V}$, $\Diamond$ ~~belong~~ pertain to a subclass of which the Post algebras are an instance.

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ENUMERABLY INFINITE VALUED FUNCTIONALLY COMPLETE PI-LOGIC ALGEBRAS

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INTRODUCTION

The subject of this article is an extension of the Pi-logic algebras to enumerably infinite case.

As shown elsewhere [1] , [2] , [3] , [4] , the Pi-logic algebras represent a class of functionally complete systems of many valued algebras of logic, of which most of the currently used systems are instances. The unifying or common properties of the Pi-algebras can be envisaged in the form of incompletely defined functions which, nonetheless, form functionally complete systems. These incompletely defined functions can be then completed in various ways, thus generating the known logic systems like that of Post, Rosser & Turquette, Zhegalkin, Aisenberg and Rabinovich etc. On the other hand they can be completed in such ways as to generate entirely new functionally complete systems of logic. This may have some importance in various applications in computing and elsewhere [4] , [5] . The systems dealt with were finite. However, some studies dealing with the sematics of stochastic and fuzzy automata [6] , [7] , [8] , which stipulate the usage of infinite valued logics, have motivated this extension of the Pi systems to the enumerably infinite case. This article presupposes the knowledge [1] , [2] , or at least the appendix of [4].

THE INFINITE VALUED PI-ALGEBRAS

Let us consider two incompletely defined and one completely defined two argument functions with arguments running through the set of non negative integers 0 , 1 , 2 , ... It is not necessary to stress that an enumerably infinite set can always be mapped into the above set of integers and thus the choice of values set can be made without losing generality.

a) $$v_1 \bigcirc v_2 = \begin{cases} v_i \text{ if } v_j = 0 \\ \text{otherwise undefined} \end{cases}$$

b) $$v_1 \square v_2 = \begin{cases} \quad \text{if } v_i = 0 \\ v_j \text{ if } v_i = 1 \\ \text{otherwise undefined} \end{cases}$$

c) $$v_1 \Leftrightarrow v_2 = \begin{cases} 1 \text{ iff } v_1 = v_2 \\ 0 \quad \text{otherwise} \end{cases}$$

Let us further introduce an enumerably infinite set of constants.

$$\{c_x\} = \{0, 1, 2, \ldots\ldots\}$$

The following one argument function

$\tilde{v} = v + 1$ (where "+" has the meaning of ordinary arithmetic addition)

We shall write $\tilde{v}^x$, thus denoting x super-positions of $\sim$ so that $\tilde{v}^x = v + x$. It is especially $\tilde{v}^o = v$. Then it is possible to formulate the following two lemmata:

LEMMA 1

$$c_{\varkappa} = \widetilde{(v \Leftrightarrow \tilde{v}^{x})}^{\varkappa} \qquad \varkappa = \{0, 1, 2, \ldots\}$$

Proof

First $(v_1 \Leftrightarrow v_2) = 0$ for any $v_1 \neq v_2$, thus as $v \neq \tilde{v}^{x}$, for whatever the substitution a for v, it is always $\tilde{v}^{x} = a + x$, where x is a deliberately chosen interger $x > 0$, and thus $a \neq a + x$, so that $(v \Leftrightarrow \tilde{v}^{x}) = 0$.

Then $\tilde{0}^{\varkappa} = 0 + \varkappa = \varkappa = c_{\varkappa}$,

so that $c_{\varkappa} = \widetilde{(v \Leftrightarrow \tilde{v}^{x})}^{\varkappa}$ Q.E.D.

Corollary

The formula given in Lemma 1 renders any constant including $c_0 = 0$, i.e. $c_0 = (v \Leftrightarrow \tilde{v}^{x})$; for generating constants $c_{\varkappa} \neq 0$ it is also possible to use the formula $c_{\varkappa} = \widetilde{(v \Leftrightarrow v)}^{\varkappa - 1}$: obviously $v \Leftrightarrow v = 1$ for any substitution of v and $1 + \varkappa - 1 = \varkappa$ for any $\varkappa \neq 0$. ($\varkappa$ an integer).

Let us now introduce the following type of characteristic function:

$$\psi_i(v) = \begin{cases} 1 & \text{iff } v = i \\ 0 & \text{otherwise} \end{cases}$$

LEMMA 2

$$\psi_i(v) = (c_i \Leftrightarrow v) = [\,(\widetilde{v \Leftrightarrow \widetilde{v}^{x}}^{\,\infty}) \Leftrightarrow v\,]$$

Proof

The correctness of Lemma 2 is obvious. The expression $(c_i \Leftrightarrow v) = 1$ iff $v = i$ and 0 otherwise, which is the definition of $\psi_i(v)$. In it's own turn $c_i = (\widetilde{v \Leftrightarrow \widetilde{v}^{x}}^{\,i})$ as shown in Lemma 1.Q.E.D.

THEOREM 1

Any function of an enumerably infinite valued logic $f(v_1, v_2, \ldots v_n) \neq 0$ can always be expressed by the following type of formula:

$$f(v_1, v_2, \ldots v_n) = \bigcirc_{\substack{\lambda=1 \\ f(a_1,a_2,\ldots a_n)\neq 0}}^{m \leq \aleph_0} [\,c_\gamma \,\square\, (\,\mathop{\square}_{\ell=1}^{\ell=n} \psi_{a_\ell}(v_\ell))\,]_\lambda =$$

$$= \bigcirc_{\substack{\lambda=1 \\ f(a_1,a_2,\ldots a_n)\neq 0}}^{m \leq \aleph_0} \left\{ (\widetilde{v \Leftrightarrow \widetilde{v}^{x}}^{\,\gamma}) \,\square\, [\,\mathop{\square}_{\ell=1}^{\ell=n} [(\widetilde{v_\ell \Leftrightarrow \widetilde{v}_\ell^{x}}^{\,a_\ell}) \Leftrightarrow v_\ell\,]_\lambda\,] \right\}$$

Proof

Allowing for the possibility that the above formula may theoretically consist of an infinite number of elements of the type $[c_\gamma \,\square\, (\mathop{\square}_{\ell=1}^{\ell=n} \psi_{a_\ell}(v_\ell)\,]_\lambda$ we can easily see the correctness of the theorem. In analogy with the finite binary case it is easy to see that any logical function can be viewed as an (ordered) set of n + 1 tuples (or vectors with n + 1 dimensions): of the type $\langle a_1, a_2, a_3, \ldots a_n; \underline{b}\rangle_\lambda$ where a_i (i = 1, 2, ...n) are respective substitutions of the variables, $v_1, v_2, \ldots v_n$ and $\underline{b}$ is the value the function assumes for that particular substitution.

Let us first consider that out of $\aleph_0$ possible vectors of the type $\langle a_1, a_2, \ldots a_n\rangle_\lambda$ only one $b \neq 0$, so that we would have a function assuming a value $b \neq 0$ just for one substitution and 0 for the rest. If we now formed a formula of the type:

$$[\psi_{a_1}(v_1) \,\square\, \psi_{a_2}(v_2) \,\square \ldots \square\, \psi_{a_n}(v_n) \,\square\, c_{\gamma}]_\lambda \quad (c_{\gamma_\lambda} = b_\lambda)$$

where each $\psi_{a_i}(v)$ is a characteristic function so that

$$\psi_{a_i}(v) = \begin{cases} 1 \text{ iff } v = a_i \\ 0 \quad \text{otherwise} \end{cases}$$

this would essentially express the function as obvious from the definition of $\psi_i(v)$. and $v_1 \square v_2$.

Let us now consider several functions of this type $\varphi_1, \varphi_2, \ldots \varphi_m$ where m is first a finite number. Obviously if $\varphi_1, \varphi_2, \ldots, \varphi_m$ were linked together with the functor $\bigcirc$, i.e. $\varphi_1 \bigcirc \varphi_2 \bigcirc \ldots \bigcirc \varphi_m$ they will express a function assuming respective values $b_{\lambda_1} \ldots \ldots b_{\lambda_m}$ just for the respective substitutions $\langle a_1, a_2, \ldots a_n\rangle_{\lambda_i}$ i = 1, 2, ...m, and 0 otherwise.

As long as the number of substitutions for which there are function values, $b_\lambda \neq 0$ is finite, there is no difficulty in envisaging the construction of the respective formula. If the number of values $b_\lambda \neq 0$ be infinite, the above type of formula could still be envisaged in the form of an infinite string of symbols. Thus any functor of an enumerably infinite valued logic $f(v_1, v_2, \ldots v_n) \neq 0$ can be expressed by the above type of formula. Q.E.D.

THEOREM 2

Any set of functions of the type $\{\bigcirc, \Box, \Leftrightarrow, \sim\}$ is functionally complete in any enumerably infinite calculus of logic.

Proof

The correctness of this theorem is obvious as the types of connectives listed are the only ones entering in the formula Theorem 1.

Corollary

If the functor $\bigcirc$ is completed in such a way so that the $v \bigcirc 1 = v + 1 = v \odot 1$, then a set of the type $\{\odot, \Box, \Leftrightarrow\}$ is also complete. It may be noted that ordinary addition is an $\odot$-type of functor. An instance of the $\Box$- type of functor is ordinary arithmetic multiplication.

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The Implementation and Evaluation of a Fuzzy Control Algorithm for a Sinter Plant

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Introduction

The reported application [1] of fuzzy logic to the control of a dynamic system has led to a study of the applicability of the technique to the design of a controller for a process where the characteristics are ill defined. It was applied to the control of the raw mix permeability in an iron ore sinter plant by controlling the rate of water addition. The moisture-permeability relationship is highly non linear and very variable.

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The Control Algorithm

The control algorithm consists of a group of rules that express the dependence of one variable upon another. The rules are expressed in terms of the fuzzy sets A,B,C and D (such as small or large) that describe the variables X Y and Z. The rules have the form:-

If X is A and Y is B or C then Z is D ... 1

where Z and Y are the inputs and Z the output of the algorithm.

From the theory of fuzzy logic [2] it is possible to show that the fuzzy set describing Z is given by:-

$$\mu_Z(w) = \underset{W}{\text{Min}} \left[\underset{U}{\text{Max}} \left(\mu_X(u) \wedge \mu_A(u) \wedge \mu_D(w) \right) , \right.$$

$$\left. \underset{V}{\text{Max}} \left\{ \mu_Y(v) \wedge (\mu_B(v) \vee \mu_C(v)) \wedge \mu_D(w) \right\} \right] \quad \ldots 2$$

where U, V and W are the universes of discourse for the variables X, Y and Z.

For a particular value of w equation 2 can be re-arranged into a simpler form:-

$$\mu_2(w) = \underset{W}{\text{Min}} \left[\mu_D(w), \left\{ \underset{U}{\text{Max}} \; \mu_X(u) \wedge \mu_A(u) \right\} , \right.$$

$$\left. \underset{V}{\text{Max}} \left(\mu_Y(v) \wedge (\mu_B(v) v \mu_C(v)) \right\} \right] \quad \ldots 3$$

Where $a \vee b = \max(a,b)$

$a \wedge b = \min(a,b)$

Definition matrices, similar to the one shown in Table I are used to define the possible fuzzy sets A,B,C etc. over the appropriate inverses of discourse

	A			
	Support	Small	Medium	Large
X	1	1	.7	.0
	2	.7	1	.3
	3	.3	.7	.7
	4	0	.3	1

Table I Definition Matrix for Fuzzy Set A on Variable X

The fuzzy sets describing each input variable X or Y have zero membership value for all but the mth support in X and the n th support in Y. This leads to further simplification of equation 3 since, for example, evaluation of the term

$$\mu = \max_{u} \left[\mu_X(u) \wedge \mu_A(u)\right]$$

reduces to the selection of an element in the column 'A' that is identified by the mth row in the definition matrix. Using matrix notation equation 3 becomes

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$$\mu_Z(w) = \text{Min}\left[W(w,D),\ U(m,A),\ \text{Max}[V(n,B),V(n,C)]\right] \quad \ldots 4$$

where U, V and W refer to the definition matrices for the fuzzy sets.

If there are J rules then they are combined using the linguistic connective 'else', ie

$$\mu_Z(w) = \underset{j=1,J}{\text{Max}}\left(\mu_{Z_j}(w)\right) \quad \ldots 5$$

Given a range of values for n and m, i.e. a range of inputs, and values (column numbers) for A B C and D equations 4 and 5 give the membership value for each support w in the fuzzy set describing the output variable Z.

Numerical values for the column numbers are obtained from an interpreter program that operates directly on each rule expressed as a test string. The rules are entered one at a time and the text string searched to identify the mnemonics used to represent the various fuzzy sets. Numerical values to identify the fuzzy sets appropriate to each rule are obtained and used to evaluate equations 4 and 5 for each input condition. The resulting fuzzy output set is converted to a crisp value by taking the support that gives the maximum membership value in the set.

This procedure has been adopted to generate a look up table employed to implement a control algorithm expressed as rules of the form shown above. This once and for all interpretation of the rules obviates the need for a resident rule interpreter in the control scheme and gives a very simple control system, although on-line modification of the rules becomes difficult.

Application to the Control of a Sinter Plant

An evaluation of the technique was undertaken on the raw mix permeability control scheme for a BSC sinter plant. Figure I shows, in block diagram form, the water addition control scheme used to optimise permeability.

The permeability error and sum of errors were used as inputs to the control algorithm. The rules defining the algorithm were written to embody experience of plant behaviour and to follow what was considered a reasonable action to take in a given set of circumstances. Examination of the initial look-up table showed some inconsistences which were resolved by adding rules and making slight changes to existing rules.

The controller was tuned by adjusting the scale factors associated with the supports of the fuzzy system variables. This was done on a simulation of the process based on the best information available.

Performance was satisfactory when process dynamic characteristics and the simulated non-linear moisture-permeability relationship were changed. Time constant changes were in the range 2:1 and gains in the range 4:1. The standard deviation of the error due to the simulated permeability measurement noise was slightly less than that obtained when a conventional two-term controller was used.

Plant Trials

The satisfactory simulation results lead to an on-plant trial at the BSC Cleveland Sinter Plant.

After start-up no further tuning of the control algorithm was required. When transients had settled it was found that the permeability standard deviation was less that that obtained under manual control.

It was also noted that control valve movements were less violent when the fuzzy logic controller was in use than when a conventional controller was completing the loop.

Further Work

The satisfactory performance of a simple fuzzy logic controller implemented as a look up table encourages further work. It is hoped to apply the method to the design of controllers for plants having non-linear dynamic characteristics and having non-linear performance criteria that are expressed as a set of heuristic relationships.

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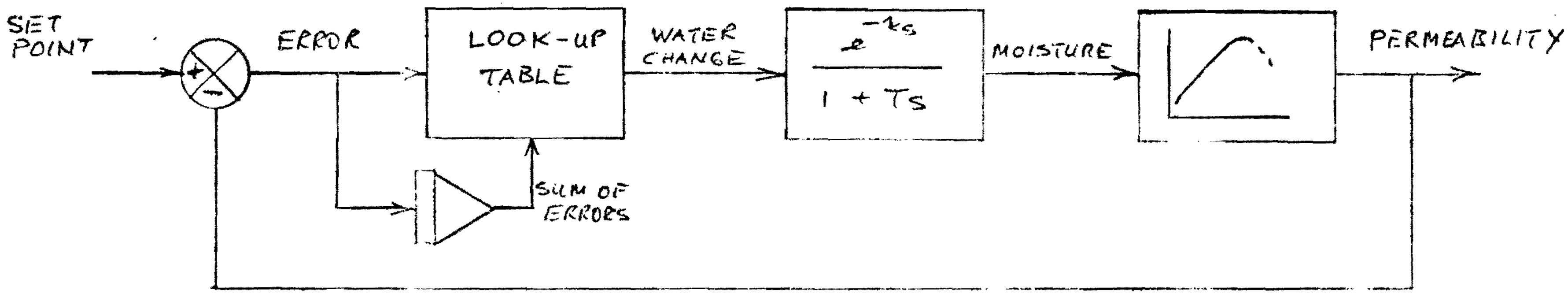

Figure 1

Block Diagram of Sinter Plant Control Scheme

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A FUZZY SET-THEORETIC APPROACH TO SEMANTIC MEMORY: A RESOLUTION TO THE SET-THEORETIC VERSUS NETWORK MODEL CONTROVERSY.

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Current models of semantic memory have been classified into set-theoretic and network models (Rips, Shoben, & Smith, 1973). The set-theoretic models (e.g., Meyer, 1970; Schaeffer, & Wallace, 1970; Smith, Shoben, & Rips, 1974a) proposethat the meanings of words are represented by sets of semantic elements or features. Network models(e.g., Collins, & Quillian, 1969; Rumelhart, Lindsay, & Norman, 1972), on the other hand, represent word meaning by interconnecting nodes with labeled relations in a network system. Hollan(1975) has argued that the Smith et al.(1974a) model is isomorphic with a network model translation and that the distinction between semantic memory models is vacuous. The present paper intends to show that the Smith et al. model can be reinterpreted in terms of a fuzzy set-theoretic (FST) model, which is a more fundamental model, and that the true distinction in semantic memory models is between network models with differing structural assumptions.

In Zadeh's (1965) formulation of the theory of fuzzy sets, each element (x) of a set(A) is characterised by a membership (characteristic) function f_x:A-(0,1) which associates with each element in A a real number in the unit interval (0,1), with the value of f_x at x representing the "grade of membership" of x in A. Smith et al.(1974b) have only introduced the theory of fuzzy sets as a means of describing the degrees of truth of propositions, an approach which is consistent with the semantic relatedness effects in sentence verification. Goguen(1967) has generalised Zadeh's system by replacing (0,1) by some more general mathematical structure, such as a completely distributive lattice or a semiring. As an extension of this generalisation, Goguen(1974) was able to show that concepts are represented by fuzzy sets by proving that the category of concepts satisfies the axioms and theorems of fuzzy sets. A FST model of semantic memory represents the meaning of a word as a set of semantic elements in which each element is graded as to its importance in the definition of the word. The FST model is equivalent to the Smith et al.(1974a) feature model in that the defining and characteristic features of a concept as specified in the feature model are the elements of the fuzzy sets. Also, the salience of a given feature to the definition of a concept is characterised by its membership function. The somewhat arbitrary partition of features into defining and characteristic can be represented as a threshold membership function; above a certain membership function value the elements are classed as defining features, and below it as characteristic features.Hence,all the structural assumptions of the Smith et al. model are satisfied, and the FST model provides a descriptive interpretation.

As Hollan(1975) proposed any set-theoretic model can be mapped onto a network model by the procedure that he describes; the FST model is no exception. The elements of the fuzzy set associated with a concept would be transformed into nodes connected to a common concept node. The membership functions would be assigned to the edges of the digraph, indicating the importance of the two nodes to each others definitions. Goguen(1974) proposed a similar model in terms of hierarchies of fuzzy sets,i.e., fuzzy sets of fuzzy sets of.... fuzzy sets, for all finite levels. As the transformed FST model is isomorphic to the Smith et al. model then the processing assumptions of the feature model should not only be capable of satisfaction within the FST model, but also within its network counterpart. The two stage model posited by Smith et al. could be realised as a mapping process in the network system(Simmons, 1973) in which the first stage is a mapping of the total concept network(i.e., defining and characteristic features) between subject and predicate nodes. The second stage would consist of the mapping of the subset of the subject concept network where the subset is defined in terms of the elements having a minimum membership function (i.e., only defining features). It is obvious that the FST model is a more fundamental description of a set theoretic approach as it does not entail any partition of features into defining or characteristic; a partition yet to be explained by Smith et al.

According to Rips et al.(1975) it is in terms of processing assumptions that the set-theoretic and network models separate, particularly the processes involved in statement verification of the form An S is a P (e.g., An apple is a fruit). The set-theoretic processes assume that the verification time is dependent upon the similarity of the subject-predicate pair by the comparison of semantic elements, while network models assume the verification time to be associated with the retrieval of pathways between the subject and predicate concept nodes. As the FST model makes the same assumptions as the feature model for both structure and process, and as the FST model also has an isomorphic network representation, then it should be possible to apply the same processes to a network representation of the FST model. If the FST model is isomorphically mapped onto a network representation then the processes required for statement verification would necessitate the following matching processes:

(a) A match between the subject and predicate networks, the extent of the networks being defined by some arbitrary minimal index or membership value for the links or edges.

(b) A short-range match over the same networks; this time the extent of the networks is defined by an arbitrary threshold membership value which partitions the network into defining and characteristic sub-networks.

Both of these matching processes can be simulated by a search through the predicate and subject networks starting at the subject and predicate concept nodes. The searches from each of the nodes would be breadth-first and the weight given to any intersections which are found is a function of the membership values attached to the links which meet at the node. This type of search is very similar to the search processes posited by the original network models of Quillian (Collins and Quillian, 1969, 1972).

In summary, by considering the Smith et al. set-theoretic model as a specific form of the general FST model, it is possible to directly translate the feature model into a network model with identical structural assumptions and similar processing assumptions. A proliferation of semantic memory models could be produced by merely starting from different terminological viewpoints. However, it is safe to say that the majority of these models would merely be different forms of the general FST model embedded in different terminology.

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Jon M.V.Slack

Possible applications of the theory of Fuzzy Sets to the study of semantic memory.

1) What is semantic memory?

Semantic memory is that part of long term memory which is involved in the comprehension of language, or to quote Tulving,

> "Semantic memory is the memory necessary for the use of language. It is a mental thesaurus, organised knowledge a person possesses about words and other verbal symbols, their meaning and referents, about relations among them, and about rules, formulas, and algorithms for the manipulation of these symbols, concepts, and relations."
>
> Endel Tulving (1972, p.386)

In the study of semantic memory we are concerned with the representation of knowledge within the human memory. This field is in many respects almost identical to the study of knowledge representation in A.I.; the study of semantic memory provides the psychological validity for the A.I. models of knowledge representation.

The various models of semantic memory which have been developed can be divided into two classes; network models and set-theoretic models. This division, however, is somewhat artificial in that the set-theoretic models can be transformed without loss of specificity, into network models which have the same structural and processing assumptions, and vice versa , the network models can be transformed into set-theoretic models. In fact, the two classes of model can be shown to be specific types of a general class of Fuzzy set-theoretic (FST) semantic memory models (paper(A)).

The main structural assumptions for the class of FST models are the following,

(a) Concepts (which are indistinguishable from concept meanings) are represented by sets of propositions of the form R(A,B) where A and B are concepts and R is the relation which holds between the concepts. Thus, the concept A can be represented as $\{R_1(A,B_1),\ R_2(A,B_2),\dots\dots\dots,R_n(A,B_n)\}$.

(b) The sets of propositions are ordered according to their importance for the meaning of the concept. Thus, if the concept A is represented as above then associated with each proposition R(A,B) there is a membership value $b_{A,B}$ which is a measure of how important the proposition R(A,B) is for the concept A.

(c) Each and every concept is represented in this way so that the concepts which are elements of a concept's set of propositions are themselves sets of propositions with other concepts as elements.

(d) The relations between concepts are themselves concepts and consist of sets of propositions, and thus R could be represented in the same way as A.

(e) The relations can be inversed, but the membership values are anti-symmetric, that is, $b_{R(A,B)} \neq b_{R^{-1}(B,A)}$.

As each and every concept is represented in this way then the 'universe of concepts' must consist of a 'universe of fuzzy sets' (Goguen, 1974).

2) Sources of Fuzziness.

Our semantic memory is built up over our lives and it is built up out of experience. Our experience, however, is continuous; experience does not arrive in little discrete packets, but flows, leading us imperceptibly from one state to another. Thus, our semantic memory is based on a continuity, and memory for what has been perceived incorporates some of this continuity. It has long been acknowledged by philosophers and more recently by psychologists and linguists that words do not have distinct, sharply delineated meanings. Wittgenstein in the Investigations expounds at length on this problem with respect to the single word 'game'. More recently, the linguist Labov(1973) has demonstrated the fuzziness of the word 'cup'. We encounter many ordinary objects that are clearly and easily named, but many more where it is difficult to say exactly what they are if we confront them directly. A moment's thought about a paradigmatic example of reference reveals that the range of applicability of a word is fuzzy. While there is universal agreement as to what is a prototypical red, it is obvious that its limits are indeterminate. One could put the question : How much can one change an object before it ceases to be the object it was? Presumably, only when it ceases to be what it was do we finally cease to call it what we did.

The above argument highlights the fact that with logic and language we are dealing with discrete symbol systems which map onto some type of continuum of concepts. Thus, communication requires us to convey what is usually some kind of continuum by using discrete symbols. Words map onto concepts but the concepts they map onto are not identical; consequently the process of mapping words onto concepts needs to be sufficiently flexible to enable the most varied members to be referenced by their proto-typical words. If there is any sense in maintaining that words have fixed meanings it can only be that independent of context they relate to their proto-typical non-linguistic counterparts. Thus, in semantic memory, two networks are required; a lexical network which stores the names of concepts and which is organised along lines of phonemic and orthographic similarity, and a semantic network which is far more complex and incorporates various continuum. The lexical network maps onto the semantic network by means of the name relation,N. In accordance with the argument outlined above, the name relation is a fuzzy relation of the form $f:N(W \times C) \rightarrow V$ where W is the set of all words, C is the universe of concepts, and V is some algebraic structure.

One of the problems alluded to in the above argument was the continuity aspect of semantic memory and one aspect of this continuity is the memory for real-world variables. Variability information must be an integral component of the memory for concepts and a necessary compoenent of memory models. In fact, any model of working memory which fails to deal explicitly with such a salient characteristic of a real-world concept as its dimensional variability is fundamentally inadequate. In traditional models of semantic memory it was assumed that physical property information is stored in discrete attribute value form, but this has been shown not to be the case (Walker, 1975). To overcome this difficulty it seems reasonable to propose a more elaborate memory model which incorporates subjective distributions of physical variability. These distributions need not be stored in semantic memory but could be generated when needed by examination of exemplars. The exemplars could be extracted from the continuous experience stored in the other areas of memory and brought into semantic memory as a fuzzy set of exemplars with the name of the prototypical object mapping onto this set. It remains to be established how much of the information in memory is stored in explicit form and how much is computed by little understood processes of fuzzy inference(Zadeh, 1972; Carbonell & Collins, 1973).

3) Fuzzy sets and the realisation of membership values.

The membership values attached to each link in the network models have been called criterialities or importance tags(Quillian, 1969; Carbonell & Collins, 1973). The psychological realisation of these values has been treated in slightly different ways. In Quillian's original theory the membership values were defined as criterialities, which are numbers indicating how essential each link is to the meaning of the concept. In Collons and Quillian(1969, 1972) links were assumed to have differential accessibility (i.e., strength or travel time). The accessibility of a proposition depends on how often a person thinks about or uses the proposition in connection with the concept. Whether criteriality and accessibility are treated as the same or different is a complex issue, but network models allow them to be treated either way. It is difficult to know whether these two terms are merely describing the same phenomena in different ways, or whether membership value is some function of accessibility and criteriality. This is essentially a psychological problem.

There is the interesting problem, untouched by Collins and Quillian, of how to model the criterialities or accessibilities. In Carbonell and Collins(1973) the importance tags were modelled by the integers from 0 to 6. The lower the tag, the more important the peice of information is. The tags add up as you go down through lower embedded levels. Thus, the criterialities are modelled by the monoid $[0,1,2,3,4,5,6,]$ with the binary operation +. Also, by using the property that the value 6 is some sort of default value, such as 'IF 6 THEN SUPPRESS PRINTING', then the monoid can be said to have an infinity element; $a + 6=6$.

In many cases the above model would be unsatisfactory, and a better model would be a multiplicative monoid such as $(J,x,1)$ where J is the unit interval $[0,1]$. This structure would be particularly suitable for the FST semantic memory model as the number propositions which comprise a concept is usually extremely large and would, therefore, require a continuous interval.

4) Context Dependency.

As stated above, it has for a long time been known that the meanings or referents of words are context dependent, and further the structure required to relate the concepts utilised in understanding language is context-dependent. Different contexts may necessitate different and in some cases even incompatible structures which cannot coexist. From this it would follow that our knowledge is not structured in a static manner but is reorganised during cognitive processing. Weinreich (1966) has argued that the sense of a word changes from sentence to sentence. Consider Weinreich's example of the verb 'to eat' in the phrases'eat steak','eat soup', 'eat an apple'. Eating an apple requires no utensil. Soup is sipped with a spoon. Eating a steak requires a knife and fork. In each case the actions of the lips, teeth and tongue are different. The general point is that a word could have different meanings in a very large number of sentences in which it might appear, even when there is some core meaning as in'eat'.

To model these sorts of dynamic processes we require a flexible, dynamic conceptual system in which the structures of the concepts are altered from context to context. In the FST model this requires a process which can remap the concept set onto the unit interval, J. Thus, the fuzzy set of propositions which represents a concept would vary with context, and over contexts (hence, over time) the concept A would be represented as $f_{C1}:C_A \rightarrow J$, $f_{C2}:C_A \rightarrow J$, ..., $f_{Cn}:C_A \rightarrow J$, where C_A is the subset of the 'universe of concepts' which holds all the propositions which forms the concept A, J is the unit interval, and f is the fuzzy set with the subscipt $C(1 \rightarrow n)$ which denotes the context. This model is similar to the time dependent fuzzy sets proposed by Lientz(1972).

Within semantic network systems, the context model described above can be realised as a plastic memorial network in which the following properties hold;

(a) The accessibility indicies which are attached to each link in the network, are a function of relevant experience.

(b) The model assumes continuous development of the accessibilites from the time of the first encoding.

(c) Besides the relatively permanent improvement, the model also assumes a temporary improvement in the accessibilities as a function of recent experience(context). This temporary improvement is achieved by means of the spreading activation model. In this model, when a concept is processed (or stimulated), activation spreads out along the paths of the network in a decreasing gradient. The decrease is inversely proportional to the accessibilities of the links in the path. Thus, the activation is like a signal from a source that is attenuated as it travels outward. The nodes and links which activated are temporarily more accessible, and thus, the structure of non-activated concepts can be adjusted by the spreading effects of activated concepts.

This type of model is still along way from being complete, but it is a step towards a context-dependent knowledge representation which has all the dynamic properties of human knowledge systems. The use of context-dependent fuzzy systems will be imperative in simulating the dynamic aspects of human knowledge.

One other concept which is potentially quite useful for the modelling of concepts or word meanings is the concept of the 'entropy' of a fuzzy set. DeLuca and Termini defined the entropy of a fuzzy set as,

$$d(f)=K\sum_{h=1}^{N} S(f(x_h))$$

where S is the function $S(x)=1 -x\ln x - (1-x)\ln(1-x)$, K is a constant, and $f(x_h)$ is the membership value for the element x_h. DeLuca and Termini(1972) show that d(f) is a measure on a psuedo-metric space with respect to the distance function $\delta(f,g)\equiv|d(f) - d(g)|$. When applied to the modelling of concepts the entropy of a concept would be a measure of how fuzzy the concept is. The fuzziness of a concept might well relate or be a function of the abstractness of the concept. Further, the entropy of a concept would vary from context to context, and so, the variance of the entropy measure over different contexts would give a measure of the flexibility of the concept. Thus, the concept of entropy would be very useful in modelling certain quantitative aspects of concepts. Such a measure could be easily incorporated into a semantic memory model.

Conclusions.

With language being a discrete symbol system and with experience being continuous in nature, it is not surprising that fuzziness runs through the whole of our language. Thus, in modelling the memory involved in the use of language it is also not surprising that any theory which deals with fuzziness is extremely useful. As the theory of fuzzy sets is developed, each new theoretical notion must be considered with respect to its possible applications for the modelling of semantic memory, and language in general.

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SUBJECTIVE EVALUATION OF FUZZY OBJECTS

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SUMMARY

This paper discusses fuzzy measures and fuzzy integrals presented by one of the authors and deals with two applications of fuzzy integrals.

Fuzzy measures are monotone set functions which are not necessarily additive. Those are defined as subjective scales for fuzziness. Fuzzy integrals are the functionals with monotonicity defined by using fuzzy measures. Those correspond to probability expectations and are discussed in comparison with Lebesgue integrals.

Application problems are concerned with subjective evaluation of fuzzy objects. One of them is the evaluation of female faces and the other is that of the residences. In those problems, a fuzzy integral model is proposed to express a man's subjective evaluation process. The model is experimentally tested.

1. INTRODUCTION

In recent years, artificial intelligence, behavioural science, and human engineering, etc. which originated in cybernetics have found many applications in all fields of engineering. Together with this tendency, a variety of problems on human subjectivity which was studied first mainly in psychology have become problems in engineering. Here, a fundamental doubt is directed toward the fact that engineering has been inquiring objectivity by eliminating subjectivity.

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Concerning subjectivity among the characteristics of men which are superior to those of machines, L. A. Zadeh presented in 1965 the concept of fuzzy sets [1], which has given a powerful means to deal with subjectivity by methods of mathematics as well as engineering. Since his proposal, fuzzy sets theory has been widely applied in the fields of automata, linguistics, algorithm, pattern recognition, decision-making, and so on.

The concept of "fuzziness" corresponding to randomness in probability theory is introduced in the fuzzy sets theory. Here, fuzziness is defined as a kind of uncertainty which is caused by subjectivity and belongs to the side of subject. On the other hand, randomness can be considered as one caused by random phenomena, i.e., objective and physical phenomena.

One of the authors has presented the concept of fuzzy measures and fuzzy integrals [2, 3, 4] which are expected to have many applications in engineering. Fuzzy measures are defined as subjective scales for fuzziness. Fuzzy integrals are the functionals with monotonicity defined by using fuzzy measures. Those correspond to probability expectations and are discussed in comparison with Lebesgue integrals.

Algebraic methods have been mainly used to approach fuzziness so far, while analytical methods have been seldom explored. Fuzzy measures and fuzzy integrals belong to analytical methods which make it possible to deal with fuzziness qualitatively and quantitatively.

Fuzzy measures are set functions with monotonicity which have not necessarily additivity, while the set functions which have been investigated in mathematics are mostly endowed with additivity such as Lebesgue measures. With this point of view, the feature of this paper will be seen where monotone set functions are studied and their applications to subjective evaluation problems are discussed.

In the applications a model of subjective evaluation on fuzzy objects is developed by using fuzzy integrals. The ability of the model is experimentally tested in two examples; one is the evaluation of female faces and the other is that of residences.

2. FUZZY MEASURES

The measures discussed so far in the theory of Lebesgue integrals or in probability theory are the set functions with additivity. Here, extending the concept of the measures, "measures" as monotone set functions which are not necessarily additive are considered. The concept of "measures" discussed in this section can be summarized in three statements concerned with grade of fuzziness.

Now, let X be an arbitrary set and ϕ an empty set. Let x denote an element of X and let A, B, etc. denote subsets of X.

First, suppose that a person picks up an element x out of X, but does not know which one he has picked up. Next, suppose that he guesses if x belongs to a given subset A. It is uncertain and fuzzy for him whether $x \in A$ or not. His guess would become subjective when there are few clues for guessing. Assume in general that a human being has a subjective quantity called the grade of fuzziness measuring fuzziness such as stated above. Then the statements are described as follows:

(1) Grade of $x \in \phi = 0$ and grade of $x \in X = 1$.

(2) If $A \subset B$, then grade of $x \in A \leq$ grade of $x \in B$.

The third statement concerned with continuity will be seen in the definition of fuzzy measures.

By the term, "the grade of fuzziness", the quantity which depends heavily on human subjectivity is implied. When a man says that an object is uncertain, two kinds of uncertainties can be considered. One is uncertainty due to the lack of information and knowledge. This uncertainty is an objective one which is characterized by the nature of objects and the circumstance surrounding them. For instance, the probability of the result of throwing a die is independent of a subjectivity and dependent only on the nature of the die and its circumstance. The other is the subjective uncertainty due to human subjectivity: the niceness of a woman's face is affected by a man's subjectivity besides her looks. The objective uncertainty is called randomness and the subjective one fuzziness.

The grade of $x \in A$ is merely an abstract example of the grade of fuzziness. As a more concrete example, "the grade of importance" stated later in the applications can be considered. Though it may be adequate for uncerstandings of the statements that the grade of importance is picked up, it is not mentioned in this section.

Now, fuzzy measures for expressing the grade of fuzziness are introduced. Let $\mathcal{B}$ be a Borel field of X. $\mathcal{B}$ has the following properties.

(1) $\phi \in \mathcal{B}$

(2) If $E \in \mathcal{B}$, then $E^c \in \mathcal{B}$.

(3) If $E_n \in \mathcal{B}$ for $1 \leq n < \infty$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}$.

[Definition 1] A set function g defined on $\mathcal{B}$ which has the following properties is called a fuzzy measure.

(1) $g(\phi) = 0$ and $g(X) = 1$.

(2) If $A, B \in \mathcal{B}$ and $A \subset B$, then $g(A) \leq g(B)$

(3) If $F_n \in \mathcal{B}$ and $\{F_n\}$ is monotone, then $\lim_{n\to\infty} g(F_n) = g(\lim_{n\to\infty} F_n)$.

Here, (1) means boundedness and non-negativity, (2) monotonicity, and (3) continuity. The property (2) is the most important one and (3) is essential only when X is an infinite set.

In the above definition g(A) is the expression of grade of $x \in A$. In general, g(A) is interpreted as a subjective measure expressing the grade of fuzziness of a set A. Of course, this does not necessarily mean that A is a fuzzy set. Though A exists objectively for any one, it is regarded fuzzy since it is associated with subjectivity when a person guesses, for instance, grade of $x \in A$. In probability theory, a set A is called an event. But the terminology "event" is not used because it is desirable to distinguish grade of fuzziness from probability (grade of randomness).

[Definition 2] $(X, \mathcal{B}, g)$ is called a fuzzy measure space.

Here g is called a fuzzy measure of measurable space $(X, \mathcal{B})$. When the domain of g is evident, g is simply called a fuzzy measure of X.

Now, additivity is the most important property among the properties of ordinary measures. It is, however, doubtful that an individual uses a "measure" with additivity when he subjectively measures fuzziness. Though a reasonable man is imagined in the theory of subjective probabilities, it would be more realistic to assume that an actual man has no additive measure, because his behaviours are often contradictory to the assumption that he uses an additive measure in evaluating things.

Monotonicity is a very natural assumption on the subjective judgements of an actual man, while additivity is a restrictive one. In many applications, it can be easily accepted that if $A \subset B$, then grade of $x \in A \leq$ grade of $x \in B$.

Further if the statements are adopted for the conditions satisfied by a man's subjective measure, it would be pointed out that the interpretation of subjective measures becomes rather free in comparison with probabilities. It is very difficult to explain, for instance, the grade of importance in terms of probabilities, which will be discussed in Section 4.

Now assuming for simplicity that X is a finite set K, a fuzzy measure of a fuzzy measure space $(K, 2^K, g)$ is constructed in the following way. Two types of fuzzy measures are proposed in this paper. Those can be easily extended to an infinite case.

Let $K = \{s_1, s_2, \cdots\cdots, s_n\}$ and

$$0 \leq g^i \leq 1, \quad 1 \leq i \leq n. \tag{1}$$

g^i is called a fuzzy density.

(A) Let

$$\frac{1}{\lambda}\left[\prod_{i=1}^{n}(1 + \lambda g^i) - 1\right] = 1, \quad -1 < \lambda < \infty. \tag{2}$$

Define for $K' \subset K$

$$g_\lambda(K') = \frac{1}{\lambda}\left[\prod_{s_i \in K'}(1 + \lambda g^i) - 1\right]. \tag{3}$$

Then g_λ satisfies all conditions of fuzzy measures. From the definition,

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it is obtained that

$$g_\lambda(\{s_i\}) = g^i, \quad 1 \leq i \leq n, \tag{4}$$

and that if $K' \cap K'' = \phi$, then

$$g_\lambda(K' \cup K'') = g_\lambda(K') + g_\lambda(K'') + \lambda g_\lambda(K')g_\lambda(K''). \tag{5}$$

When $\lambda = 0$, g_λ becomes additive and, hence, equal to a probability measure. g_λ is called type A. It follows from Eq. (5) that if $\lambda \leq 0$, then

$$g_\lambda(K' \cup K'') \leq g_\lambda(K') + g_\lambda(K''), \tag{6}$$

and if $\lambda > 0$, then

$$g_\lambda(K' \cup K'') > g_\lambda(K') + g_\lambda(K''). \tag{7}$$

(B) Let

$$(1-\lambda) \bigvee_{i=1}^{n} g^i + \lambda \sum_{i=1}^{n} g^i = 1, \quad 0 \leq \lambda \leq 1.^{\dagger} \tag{8}$$

Define for $K' \subset K$

$$g_\lambda^*(K') = (1 - \lambda) \bigvee_{s_i \in K'} g^i + \lambda \sum_{s_i \in K'} g^i. \tag{9}$$

When $\lambda = 1$, g_λ^* becomes additive. There holds, if $K' \cap K'' = \phi$,

$$g_\lambda^*(K' \cup K'') \leq g_\lambda^*(K') + g_\lambda^*(K''). \tag{10}$$

This is called type B.

3. FUZZY INTEGRALS

In this section, fuzzy integrals are defined by using fuzzy measures shown in Definition 1.

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† $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, $\bigvee_{i=1}^{n} a_i = \max_{1 \leq i \leq n} \{a_i\}$.

[Definition 3] Let $h : X \to [0, 1]$ be $\mathcal{B}$-measurable function. A fuzzy integral over A is defined in the following form.

$$\int_A h(x) \circ g(\cdot) = \sup_{\alpha \in [0,1]} [\alpha \wedge g(A \cap F_\alpha)],$$

where $F_\alpha = \{x | h(x) \geq \alpha\}$.

In the above definition, the symbol $\int$ is an integral with a small bar and also shows a symbol of the letter f. The small circle is the symbol of the composition used in the fuzzy sets theory.

Hereafter, it is assumed that all functions discussed in this paper, including constants, have the range [0,1]. For simplification, a fuzzy integral is written as $\int_A h \circ g(\cdot)$ or $\int_A h \circ g$. In the case of $A = X$, it is written briefly as $\int h \circ g$. Fuzzy integrals have the following properties.

Let $a \in [0,1]$, then

$$\int a \circ g(\cdot) = a, \tag{11}$$

$$\int (a \vee h) \circ g(\cdot) = a \vee \int h \circ g(\cdot), \tag{12}$$

$$\int (a \wedge h) \circ g(\cdot) = a \wedge \int h \circ g(\cdot). \tag{13}$$

If $h \leq h'$, there holds

$$\int h \circ g(\cdot) \leq \int h' \circ g(\cdot). \tag{14}$$

If $A \subset B$, then there holds

$$\int_A h \circ g(\cdot) \leq \int_B h \circ g(\cdot). \tag{15}$$

If $\{h_n\}$ is a monotone sequence of $\mathcal{B}$-measurable functions, then

$$\lim_{n\to\infty} \int h_n \circ g = \int \lim_{n\to\infty} h_n \circ g. \tag{16}$$

If $\{h_n\}$ is a monotone decreasing (increasing) sequence of $\mathcal{B}$-measurable functions and $\{a_n\}$ is a monotone increasing (decreasing) sequence of real numbers, then

$$\int [\bigvee_{n=1}^{\infty} (a_n \wedge h_n)] \circ g = \bigvee_{n=1}^{\infty} [a_n \wedge \int h_n \circ g]. \tag{17}$$

There holds $\int_A h \circ g = M$ if and only if $g(A \cap F_M) \geq M \geq g(A \cap F_{M+0})$, where $F_M = \{x | h \geq M\}$ and $F_{M+0} = \{x | h > M\}$.

The fuzzy integrals are very similar to the Lebesgue integrals in their definition. Let h(x) be a simple function such that

$$h(x) = \sum_{i=1}^{n} \alpha_i \chi_{E_i}(x)^{T}, \tag{18}$$

where $X = \sum_{i=1}^{n} E_i$, $E_i \in \mathcal{B}$, and $E_i \cap E_j = \phi (i \neq j)$.

In the measure space $(X, \mathcal{B}, \mu)$, the Lebesgue integral of h over A is defined as

$$\int_A h \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A \cap E_i). \tag{19}$$

Here assume $0 \leq \alpha_i \leq 1$ $(1 \leq i \leq n)$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Let further $F_i = E_i + E_{i+1} + \cdots + E_n$ $(1 \leq i \leq n)$. Then a simple function h(x) can be also written as

$$h(x) = \bigvee_{i=1}^{n} [\alpha_i \wedge \chi_{F_i}(x)], \tag{20}$$

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^{T} $\chi_E(x) = 1$ if $x \in E$ and $\chi_E = 0$ if $x \notin E$.

and two expressions are identical. With respect to a simple function h on X, there holds

$$\int_A h \circ g(\cdot) = \bigvee_{i=1}^{n} [\alpha_i \wedge g(A \cap F_i)]. \tag{21}$$

The similarity of Lebesgue and fuzzy integrals is clarified by comparing Eq. (18) with Eq. (20) and Eq. (19) with Eq. (21), respectively.

Next a quantitative comparison is tried. Let h be a $\mathcal{B}$-measurable function. Then both integrals, fuzzy and Lebegue, with respect to a probability measure P can be defined and the following inequality is obtained. Let $(X, \mathcal{B}, P)$ be a probability space and $h : X \to [0,1]$ be a $\mathcal{B}$-measurable function, then there holds

$$\left| \int_X h(x)\, dP - \int_X h(x) \circ P(\cdot) \right| \leq \frac{1}{4}. \tag{22}$$

Since the operations of fuzzy integrals include only comparisons of grades, the above inequality implies that using only $\vee$ and $\wedge$, a value different by at most 1/4 from a probabilistic expectation can be obtained.

A fuzzy integral in $(K, 2^K, g)$ is calculated as follows.
Let $h : K \to [0,1]$. Assume $h(s_1) \leq h(s_2) \leq \cdots\cdots \leq h(s_n)$. If not, rearrange in an increasing order.
Define

$$K_i = \{s_i, s_{i+1}, \cdots\cdots, s_n\}, \quad 1 \leq i \leq n. \tag{23}$$

Then it is obtained from Definition 3 that

$$\int_K h(s) \circ g(\cdot) = \bigvee_{i=1}^{n} [h(s_i) \wedge g(K_i)]. \tag{24}$$

There exists at least one j such that

$$h(s_{j-1}) \wedge g(K_{j-1}) \leq h(s_j) \wedge g(K_j), \qquad (25)$$

$$h(s_j) \wedge g(K_j) \geq h(s_{j+1}) \wedge g(K_{j+1}). \qquad (26)$$

Clearly, there holds for this j

$$\oint_K h \circ g = h(s_j) \wedge g(K_j). \qquad (27)$$

Thus, the value of a fuzzy integral is obtained without evaluating $h(s_i) \wedge g(K_i)$ for all i's. More precisely, it is necessary for the calculation of a fuzzy integral to evaluate $g(K_i)$ at least for only three different i's. This fact is a very excellent one in comparison with ordinary integral calculus.

A fuzzy integral is also called a fuzzy expectation in the sense of comparing it with a probabilistic expectation. As can be clarified from the preceding discussions, the essential difference between a probabilistic quantity and a fuzzy one is that the former has additivity while the latter has only monotonicity. Therefore the meaning of difference between "randomness" and "fuzziness" can be grasped through the difference between a probability measure P and a fuzzy measure g.

As is well known, the essential property of ordinary integrals is additivity stating that the area of a figure consisting of a triangle and a square equals the area of the triangle added by that of the square. Apart from visual figures such as a triangle or a square, "area" in a mathematically abstracted world is something with additivity hidden behind objects. This "area" can be measured by means of integrals which are constructed by measures with additivity. Thus it is possible to state that measures with additivity are used to measure quantities with additivity and also suitable for this purpose.

Now assume that objects have not additivity but at least monotonicity. By what means can such objects be measured? Of course it may be possible to measure those by the ordinary measures. But their additivity does not seem to suit the objects which have no additivity. The fuzzy measures introduced in this section have monotonicity but not always additivity. Is it not expected that fuzzy measures are more suitable than ordinary ones to measure the objects with only monotonicity?

4. APPLICATIONS

In this section, fuzzy integrals are applied to the problems of subjective evaluation of fuzzy objects. Fuzzy measures, as has been discussed in Section 2, are considered as subjective measures for grade of fuzziness. When application problems are discussed, however, it is convenient to interpret fuzzy measures more concretely. This will be discussed in the examples of applications in this section.

Now when a human being tries to measure and evaluate the objects which seem fuzzy, his evaluation is related to both, the nature of the objects and his own subjectivity. In general, there appears in the process of subjective evaluation the complicated interplay between the objects and the evaluater's subjectivity. In this sense, fuzzy measures should be considered to change their properties affected by the both of the objects and his subjectivity.

The evaluation problems treated so far in systems engineering are mostly those which are based on objective standards, e.g., the performance indices of optimal control systems. However, the evaluation problems are discussed here which lead to different results according to the subjectivities of the individuals evaluating the objects. The concept of fuzzy measures is powerful particularly for dealing with these problems.

4.1 Subjective evaluation of female faces

Here, the problem of the evaluation of female faces is discussed. Pictures of about 100 young ladies were taken. The boundary conditions of these pictures are kept constant carefully. Thirty pictures are chosen at random and enlarged to actual size. Each of these pictures is cut into five pieces; those are eyes, nose, mouth, chin and all the remains, as are shown in Figs. 1 and 2. Those pieces are shown to a student (male) separately and according to his preference they are scored with a numerical value between zero and one. The ideal face is scored one and the worst is zero. Now five values are obtained for each face. Next the complete picture is shown to the student, who is asked to score it by the same scoring rule. The problem is how to connect the score of a whole face with those of pieces.

Generally, when a system is perfectly decomposed into mutually independent factors, a linear model is usually used to relate the overall and the partial evaluations. However, if the boundaries among the factors are not sharp and the factors influence each other, a fuzzy integral model is one of the powerful means to evaluate such fuzzy objects.

The symbols s_1, s_2, $\cdots$,s_5 are used for eyes, nose, mouth, chin and the remains.

Define

$$K = \{s_1, s_2, s_3, s_4, s_5\}.$$

From the above experiments, the function

$$h_j : K \to [0, 1] \quad (28)$$

is obtained where j is the number of pictures.

If a linear model is used, the preference w_j of j-th face is expressed as follows.

$$w_j = \sum_{i=1}^{5} a_i \, h_j(s_i) \quad (29)$$

Using the fuzzy measure g_λ (type A) which is a student's subjective scale concerned with grade of importance, a fuzzy integral model is introduced as follows. Here, grade of importance means to what extent one attaches importance to the elements of a face.

Define

$$e_j = \oint_K h_j(s) \circ g_\lambda(\cdot). \quad (30)$$

Let $\bar{e} = \max_{1 \leq j \leq N} \{e_j\}$ and $\underline{e} = \min_{1 \leq j \leq N} \{e_j\}$, where N is the total number of faces. Let d_j denote the score of the whole face which is obtained from the experiment. Similarly, $\bar{d}$ and $\underline{d}$ are defined. Now, e_j is normalized so that $\bar{e} = \bar{d}$ and $\underline{e} = \underline{d}$. The preference w_j is obtained as follows.

$$w_j = \frac{\bar{d} - \underline{d}}{\bar{e} - \underline{e}} e_j + \frac{\underline{d}\bar{e} - \bar{d}\underline{e}}{\bar{e} - \underline{e}} \quad (31)$$

The fuzzy measure g_λ is identified so as to minimize the following criterion J.

$$J = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (d_i - w_i)^2} \quad (32)$$

When "complex method" was used for hill-climbing, the minimum value of J was about 0.1.

The comparison of the calculated value w with the experimental one d is shown in Fig. 3. Fig. 4 shows the fuzzy measures of two students. In this figure, if g^i for a specific i is larger than the others, it means that the student thinks the i-th piece very important. So it is possible to know from Fig. 4 the characteristics of an individual who evaluates ladies' faces. As is shown in Fig. 3, the experimental results show a good agreement with the calculation by the model.

The process of subjective evaluation can be explained qualitatively by using the concept of fuzzy measures. When a linear model is adopted, it is difficult to interpret the weighting coefficients. In Eq. (29), if a coefficient a_i is large, then a partial evaluation $h(s_i)$ is enlarged. This implies that the value of the overall evaluation increases in the linear model even if a partial evaluation is small. However, a man will give actually a relatively small value to the overall evaluation when $h(s_i)$ is small.

On the contrary, in the fuzzy integral model, it can be approximately stated that if a partial evaluation $h(s_i)$ is smaller than a fuzzy density g^i, then $h(s_i)$ contributes directly the overall evaluation and if $h(s_i)$ is larger than g^i, then the value of $h(s_i)$ is cut at that of g^i. This implies that a large value of i-th partial evaluation is cut when the grade of importance of i-th element is small. Therefore, it could be said that a fuzzy integral model can explain a human evaluation process more qualitatively than a linear model. Further it should be pointed out that the concept of the grade of importance is convenient in representing a subjective evaluation process.

4.2 Subjective evaluation of residences

This section applies a fuzzy integral model to represent the sujective evaluation process of residences. A residence can be decomposed into four factors such as Facilities and furniture, Area, Time from residence to office, and Environment. These factors are important when the functions of a residence from the physical, psycological and physiological aspects of human life are considered. Financial aspects such as the price of a residence and its maintenance cost are excluded in this evaluation, because this sort of factors are not regarded as the constraints related directly to the fineness of a residence.

A man's preference for a residence can be expressed with a relation between the fineness of these four factors and the grade of the importance which he attaches to each of them.

Let

$$K = \{s_1, s_2, s_3, s_4\}, \tag{33}$$

where s_1, s_2, s_3 and s_4 show F, A, T and E, respectively.

The value assigned to the fineness $h(s)$ of a factor s is determined according to the common sense; it is not the value experimentally obtained. Let $h_F = h(s_1)$, $h_A = h(s_2)$, $h_T = h(s_3)$ and $h_E = h(s_4)$. Then $h(s_i)$ for each residence is calculated as follows.

$$h_F = 0.1a + 0.05b + 0.1c + 0.2d + 0.1e + 0.1f$$
$$+ 0.2g + 0.1h + 0.05i, \tag{34}$$

where the variables at the right side are the fineness of the facilities for heating, cooling, water supply, drains, toilet, gas, bath and garrage, respectively.

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$$h_A = 0.8j + 0.2k, \quad (35)$$

where j and k are the degree of satisfaction obtained from the floor and the garden space, respectively.

For example, j and k are given as shown in Fig. 5 when a family is consisted of a couple and two children.

$$h_T = 0.5 + \frac{1.42}{\pi} \tan^{-1} 0.05(60 - \ell), \quad (36)$$

where ℓ is the total time required to go to an office from the residence.

The graph of h_T is shown in Fig. 6.

$$h_E = 0.2m + 0.2n + 0.1o + 0.1p + 0.1q + 0.1r + 0.1s + 0.1t, \quad (37)$$

where the variables at the right side correspond to cleanness of air, disturbance by unpleasant noise, fineness of sun shinning, convenience for shopping, playing, going to school and hospital, and green area, respectively.

All the samples of the residences used in the experiment are roughly scored by assingning the value between zero and one to each variable from a to t. The values for the aggregated four factors $h(s_i)$ for $1 \leq i \leq 4$ are shown in Table 1.

In this application, a fuzzy measure g_λ^* (type B) is adopted as a subjective measure for preference. According to Eqs. (30) and (31), a fuzzy integral model of preference is obtained. In analogous way of the previous section, several subjects are asked to score each residence with a numerical value between zero and one according to their preference. Then their fuzzy meausres are identified as in Section 4.1.

A linear model of Eq. (29) is also examined in this case and the weighting coefficients are determined by the same method.

The experimental results for a house wife are shown in Table 2. From these, it may be concluded that (1) environment is attached high importance, (2) the importance of facilities and that of area are medium, and (3) time is ignored. In her case, almost same conclusions are obtained by the both methods.

Note that in the both applications, the identified fuzzy measures do not satisfy additivity.

CONCLUSIONS

In this paper, the concept of fuzzy measures and fuzzy integrals has been presented and two applications have been discussed. A fuzzy measure is a monotone set function and it is regarded as a subjective scale for fuzziness. A fuzzy integral represents the value of subjective evaluation measured by a fuzzy measure.

In the evaluation problems of female faces and residences, a fuzzy integral model has been proposed to express a man's subjective evaluation process. Its effectiveness has been also clarified.

It is expected that the idea of fuzzy measures and integrals will be widley applicable in the many fields of engineering.

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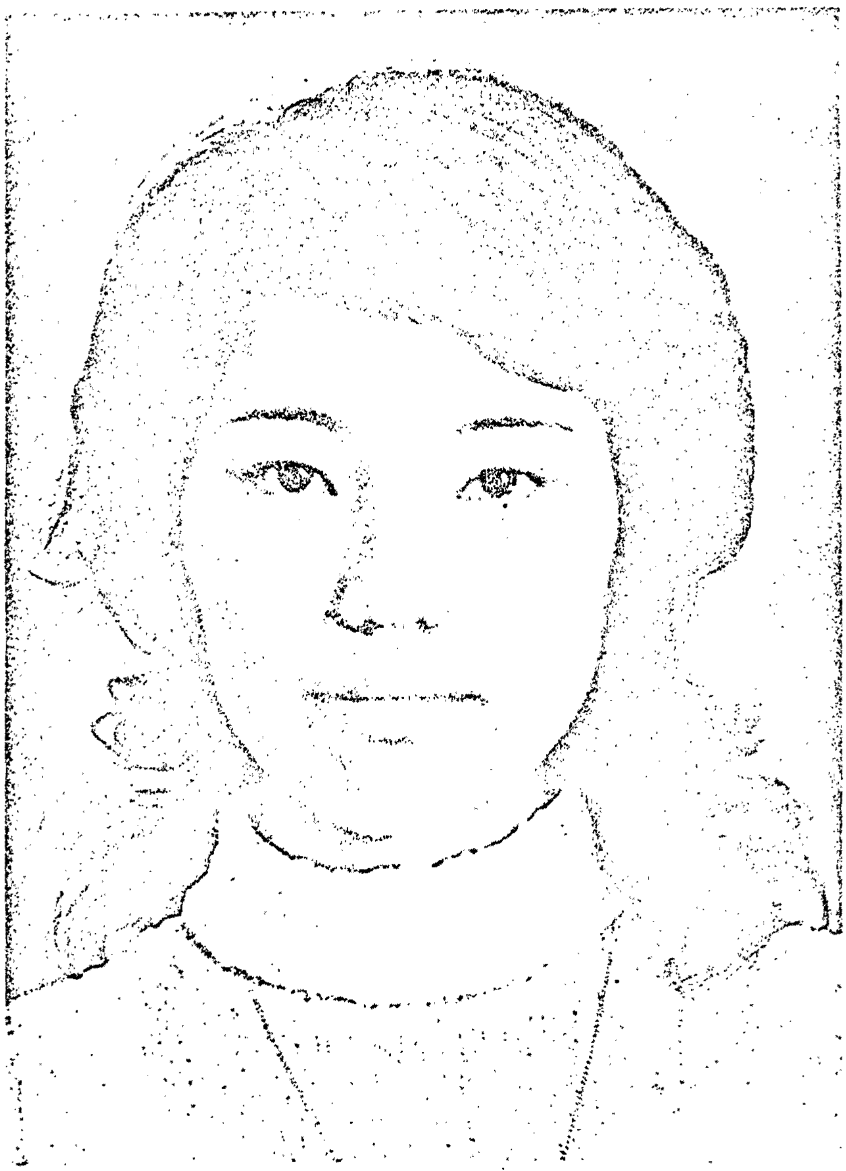

Fig. 1 Example of Female Face

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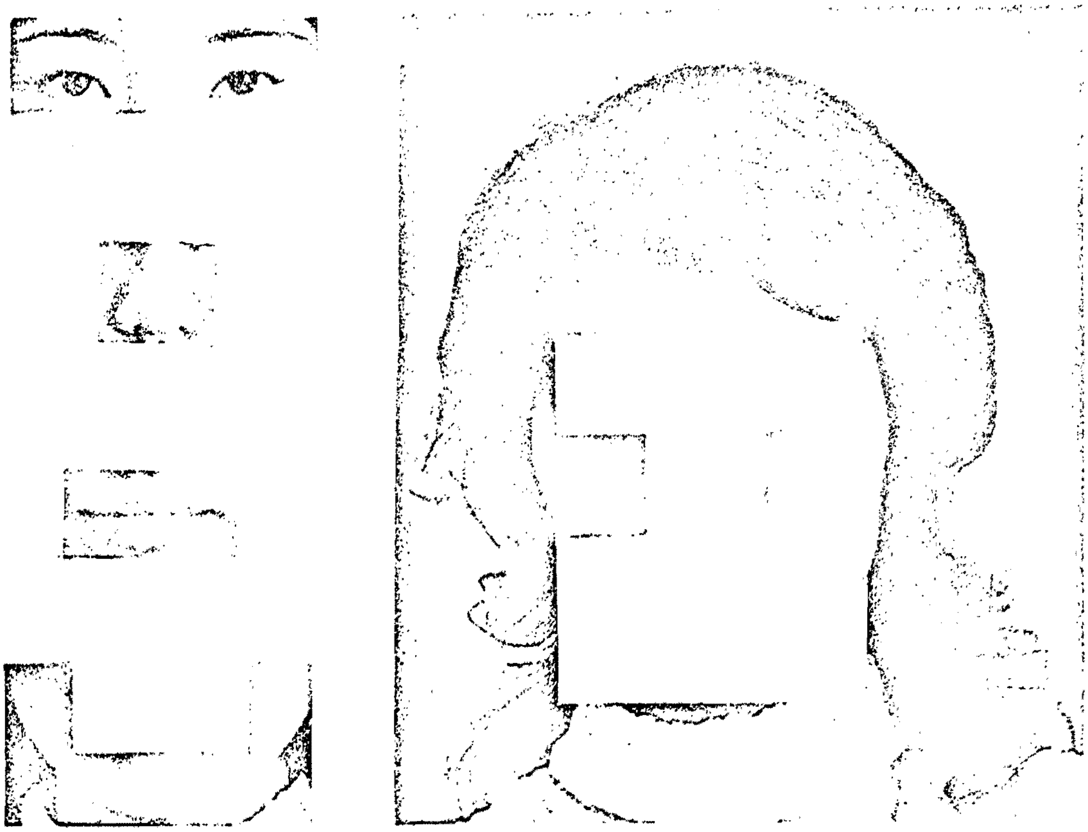

Fig. 2 Elements of Face

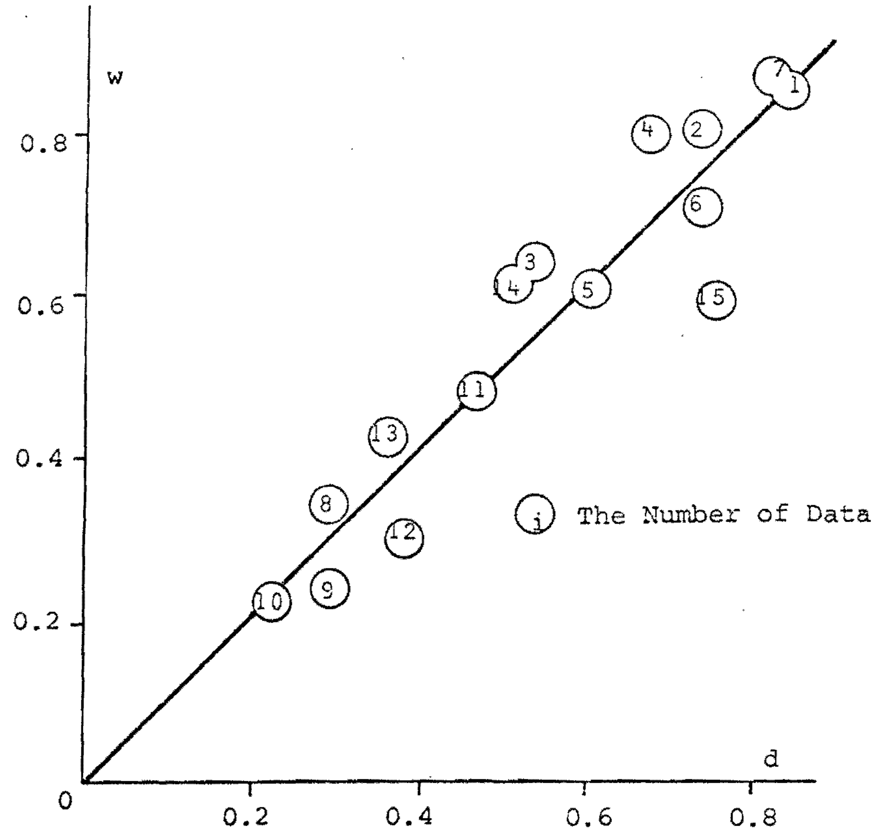

Fig. 3 Comparison of w(model) and d(experiment)

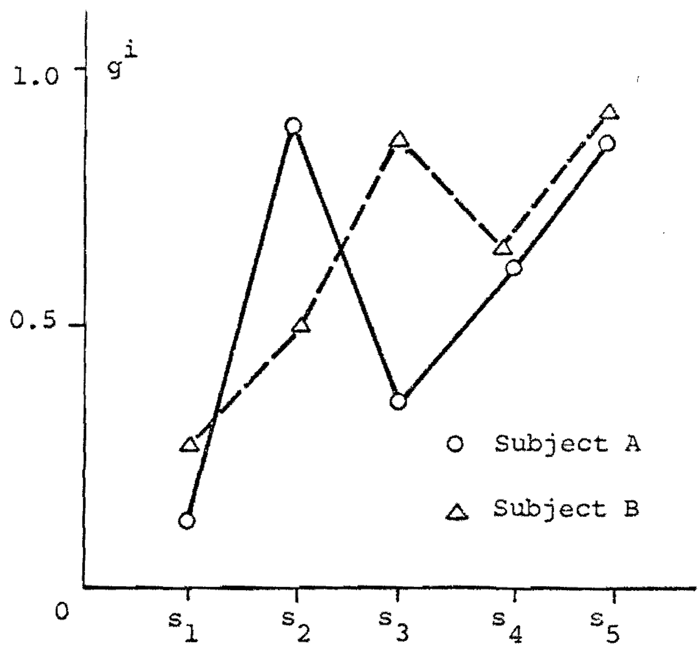

Fig. 4 Identified Fuzzy Densities of Subjects A and B

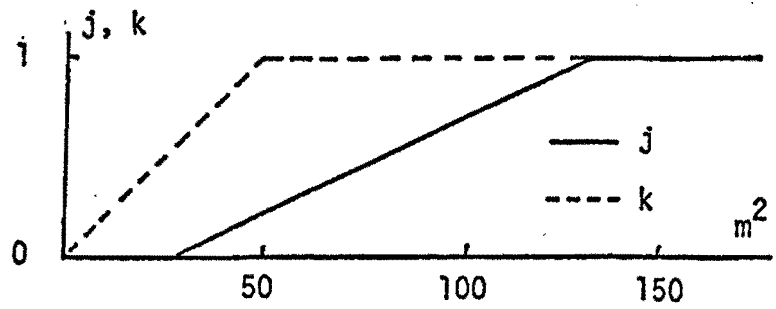

Fig. 5 Grade of Ideal Space of Floor and Garden

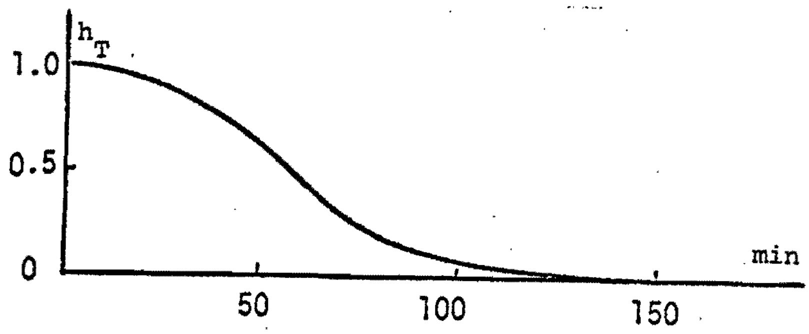

Fig. 6 Grade of Ideal Time from Residence to Office

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The Number of Residences	h_F	h_A	h_T	h_E
1	0.86	0.73	0.79	0.53
2	0.84	0.57	0.85	0.75
3	0.56	0.26	0.39	0.00
4	0.87	0.50	0.61	0.68
5	0.60	0.12	0.94	0.63
6	0.74	0.80	0.21	0.50
7	0.97	0.92	0.61	0.90
8	0.66	0.41	0.85	0.68
9	0.85	0.49	0.29	0.73
10	0.72	0.51	0.15	0.78
11	0.51	0.31	0.39	0.48
12	0.61	0.10	1.00	0.00
13	0.97	1.00	0.94	0.80
14	0.68	0.32	1.00	0.45
15	0.62	0.42	0.21	0.78
16	0.64	0.76	0.15	0.80
17	0.90	0.84	0.50	0.83
18	0.71	0.61	0.03	0.80
19	0.95	0.67	0.97	0.70

Table 1 Fineness of Residences

	F	A	T	E
Coefficient of Linear Model	0.78	0.43	0.17	1.0
Fuzzy Density	0.62	0.78	0.33	1.0

Table 2 Grade of Preference for Residences

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An assessment of a fuzzy control algorithm for a non-linear multi-variable system

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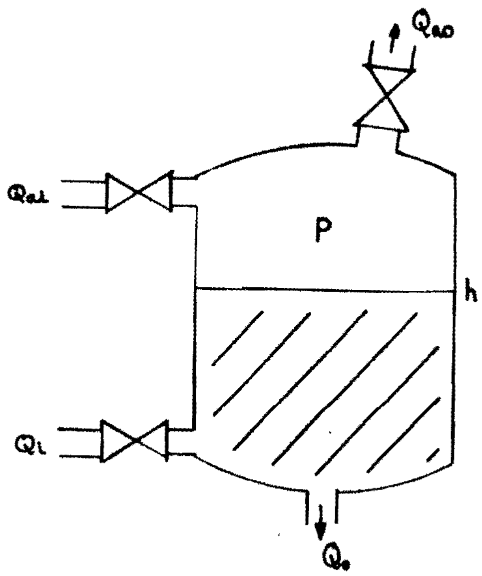
1. Introduction

Several important industrial processes, in particular the basic oxygen steelmaking process, cannot be satisfactorily controlled by the use of standard control theory. The reasons for this are several, but some of these are the lack of quantitative information about the process which is invariably non-linear and multi-variable, and the need to interface meaningfully with the process operator. On the other hand, there is often considerable qualitative information in the form of standard operator practices which reflect experience and training. Earlier workers, Mamdani and King [1] and Rutherford [2] , have demonstrated the usefulness of Zadeh's notion of fuzziness [3] in the design of control algorithms based on logical statements about linguistic variables, and this paper presents some further results from a preliminary study into the design of fuzzy controllers for non-linear, multi-variable systems.

2. Simulation experiment

A moderately difficult plant to control is a pressurised tank containing liquid. This can be represented schematically as,



where h is the liquid level inside the tank, P is the internal air pressure, Q_i is the liquid inflow and Q_{ai} is the air inflow. The control problem is to regulate both the liquid level and the total pressure inside the vessel. The tank's behaviour is governed by two nonlinear differential equations, namely

$$\frac{dh}{dt} = a_0 H^{\frac{1}{2}} + a_1 Q_i$$

$$\frac{dH}{dt} = b_0 (H-h)^{\frac{1}{2}} + b_1 Q_{ai} + b_2 (1 + b_3 (H - h))(Q_i - b_4 H^{\frac{1}{2}})$$

where H is the total pressure (h + P) and a_0, a_1, b_0, ..., b_4 are constants which depend upon the physical properties of the tank.

One advantage of using this process for a preliminary study is that it has been considered in the control literature. In particular, Macfarlane and Belletrutti [4] have linearised these equations and designed a controller using characteristic locus methods. Further, using their linearised equations an optimal stochastic regulator [5] can be designed against which the fuzzy controller can be compared.

The pressurised tank, called a headbox in certain paper-making processes, has four characteristic features which are important in designing a fuzzy controller. These are -

a. the time constant associated with total pressure changes is much faster than that associated with changes in liquid level

b. both inputs affect total pressure, but only air inflow affects liquid level significantly

c. a positive change in air inflow produces a negative change in liquid level

d. the process is stable for small perturbations about the operating point

3. Algorithm structure and design

Because the headbox has two very different time constants, the algorithm concentrates on bringing the liquid level to the set point before attempting control of total pressure. It does, though, try to minimise the change in total pressure which results from control action designed to regulate liquid level. Regulation of total pressure is only attempted when the liquid level is approximately at its set point.

Since air inflow is the only input which alters liquid level, it is used as the main control variable. Because airflow also changes total pressure, the other input, liquid inflow, is used to counteract these changes. When liquid level is under satisfactory control, liquid inflow becomes the main control variable since it is the only one which can affect total pressure.

Another general feature of the algorithm is that, since the process is stable about the operating point, in situations where the control policy is not obvious the algorithm makes no change to the controller output.

Following Mamdani and King, the inputs to the algorithm are error and change in error but in contrast to them, and Rutherford, the outputs from the algorithm can be either absolute values or incremental values depending upon the size of the error. If the error is "large", then the outputs take absolute values, and only when the error is "small" does the algorithm give incremental control outputs.

To aid comparison with previous work, the fuzzy sets have been given similar names to those used by Mamdani and King, and some of the rules used are shown below.

IF e_H = ANY . de_H = ANY . e_h = PM + PS . de_h = PM THEN dQ_i = NM . dQ_{ai} = PM

IF e_H = PM+PS . de_H = ANY . e_h = ZE . de_h = ZE THEN dQ_i = NS . dQ_{ai} = ZE

IF e_H = $\overline{NB + PB}$. de_H = ANY . e_h = PB . de_h = ANY THEN Q_i = NB . Q_{ai} = PM

where e_H, e_h are the pressure error and level errors respectively, de_H, de_h are the changes in the errors and Q_i and Q_{ai} are the liquid and air inflows. The complete algorithm has 37 incremental rules and 9 absolute rules. It is not necessarily the "best" controller for this process but is a useful vehicle for experimental purposes.

4. Algorithm implementation

As pointed out in earlier work, it is possible to "tune" the rules at several levels. Firstly at the level of the set definition, secondly at the level of the support set definition, and finally at the level of the rules themselves. Undoubtedly, the latter is more powerful but, in engineering situations, changing the support sets is probably easier since it is equivalent to changing the loop gains of the controlled process. The primary aim of this study, then, was to assess the sensitivity of the algorithm to changes in its implementation. A secondary aim was to observe the performance of the controller in a noisy environment.

The support set for error and change in error was the set of real numbers which was divided into seven discretised levels, namely

Level	Range
1	$e < -0.5$
2	$-0.5 \leqslant e < -0.2$
3	$-0.2 \leqslant e < -0.1$
4	$-0.1 \leqslant e < 0.1$
5	$0.1 < e \leqslant 0.2$
6	$0.2 < e \leqslant 0.5$
7	$0.5 < e$

The support set for the control output was also the set of real numbers but was approximated by five discrete points, namely

$$U = \{-10.0, -1.0, 0.0, 1.0, 10.0\}$$

The single-valued measurements of the process were considered to be fuzzy singletons and the output control set was derived by using the compositional rule of inference. The single-valued control action was chosen from the control set by selecting that with the maximum membership function value.

A diagram of the simulation configuration is shown in figure 1. An amplitude constraint is imposed on the control inputs to the process thus simulating a control valve which is either fully open or fully closed.

The results of the next section show the effects of changing the range of the control outputs from the incremental rules in the algorithm by changing gains G and G . They also demonstrate the effect of changing the process sampling interval and the effect of adding noise to the measurements.

5. Simulation results

The simulations consist of five experiments. The first three of which investigate the response of the process to changes in the set points. The fourth experiment compares these responses with those of the controller designed by Macfarlane and Belletrutti. The final experiment compares the performance of the fuzzy algorithm with both Macfarlane and Belletrutti's controller and an optimal stochastic controller when noise is present in the measurements. Not all the results of these experiments are shown here but can be found in reference [6] which has full details of the control rules.

5.1 First experiment : set point change

The first experiment examines the response of the process when both set points are increased by +1.0. Figure 2 shows the response when $G_1 = G_2 = 0.1$, the sampling interval is 0.1s and there is no noise present. In common with all the results shown in this section, the zero line is the nominal steady state of the process. This corresponds to a liquid level of 3 ft., a total pressure of 16 ft. of water, a liquid inflow of 24 cu.ft./s and an air inflow of 26 cu.ft./s.

Notice how, in figure 2, the characteristic features of the algorithm are exhibited. The controller works hard to bring liquid level near its set point. Only when this is achieved is direct control of total pressure attempted. Notice also the two phases of the controller. In the first three seconds when the error is large, the algorithm gives absolute values of control, thereafter, the error is small and the algorithm has an incremental output.

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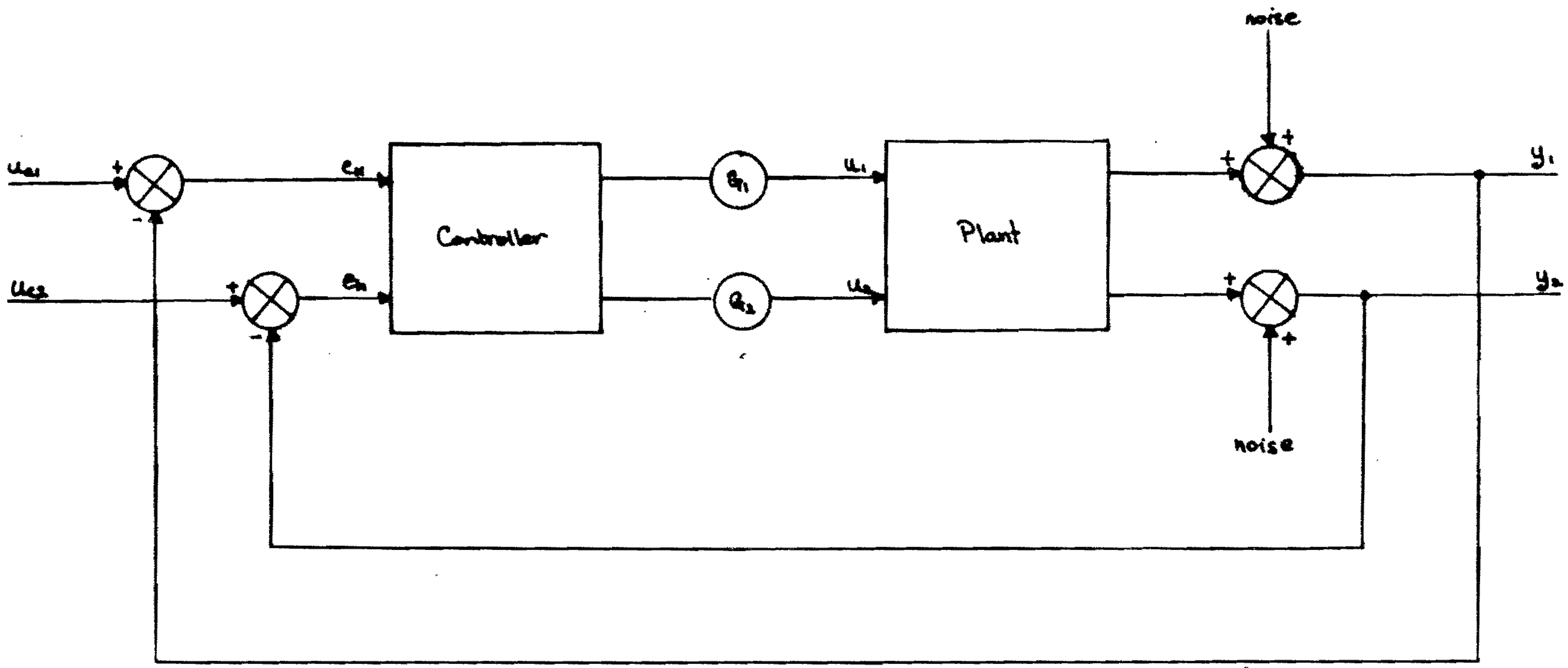

Figure 1.

The controlled response is therefore seen to be quite good. The open loop-time constant of liquid level is of the order of ten minutes whereas, using the algorithm described, liquid level is brought to its set point within 20 seconds under closed loop control.

Figure 3 shows the effect of setting $G_1 = G_2 = 1.0$ but leaving the sampling interval unchanged. Notice how the response is improved, liquid level reaching set point, with no overshoot, in under ten seconds. This is achieved, though, at the expense of more vigorous control action.

Figure 4 shows the effect on this response when the sampling interval is increased to 2.0s. In many ways, the response is better, but, as the results of the next section show, this is not a consistent result.

5.2 Second experiment : set point change

This experiment demonstrates the response of the process to a change in set point of +1.0 in total pressure only. Figure 5 shows the behaviour when $G_1 = G_2 = 0.1$ and the sampling interval is 0.1s. Notice the characteristic limit cycle of a non-linear system. Increasing the gains so that $G_1 = G_2 = 1.0$ has the desired effect, see figure 6, of bringing total pressure to its new set point in about 12 seconds without disturbing liquid level significantly. However, increasing the sampling to 2.0s gives rise to oscillatory behaviour, see figure 7.

5.3 Third experiment : set point change

The final set point experiment examines the systems response to a set point change of +1.0 in liquid level only.

Figure 8 shows the response when $G_1 = G_2 = 0.1$ and the sampling interval is 0.1s. Note the limit cycles. In this case changing G_1 and G_2 to 1.0 was not sufficient to stabilise the response but was only achieved by setting $G_1 = 2.0$ and $G_2 = 1.0$, see figure 9. Again, increasing the sampling interval, this time to 0.5s, gives an unsatisfactory response, figure 10.

5.4 Fourth experiment : comparison with 'M & B' controller

In order to compare the best responses of the fuzzy controller with a controller designed in a more conventional way, it is necessary to linearise the equations given in section 2. This has been done by Macfarlane and Belletrutti who also designed a controller. Simulation of the process with their controller replacing the fuzzy controller gave the results shown in figures 11, 12 and 13, corresponding to set point changes in pressure and level, pressure only and level only. These are much better, as expected, although the overall shape of the controller response in each case is not dissimilar to that of the fuzzy controller.

5.5 Fifth experiment : stochastic control

Using the linearised headbox equations of the previous experiment, it is possible to design an optimal stochastic regulator for the headbox. Figures 14, 15, 16 and 17 show the behaviour of the controlled process when measurement noise is introduced to the system. This experiment consisted simply of keeping the set points at their set state values and altering the variance of the measurement noise. The mean square output values and the mean square control values were calculated for each combination of noise variance and controller and are plotted in the figures.

Figure 14 shows the mean square total pressure, $\overline{y_1^2}$. Notice how close are the curves of the optimal stochastic and M & B controller. But notice also the shape of the fuzzy controller curve. For a range of noise variances between 0.01 and 0.1 this is almost flat, indicating that the controller is insensitive to noise amplitude over this range. Figure 15 shows the same curves for mean square liquid level, $\overline{y_2^2}$. In this case all three controllers give a similar performance.

Figure 16 and 17 show the mean square controls, $\overline{u_1^2}$, and $\overline{u_1^2}$, and give a rather different picture. Whilst the optimal stochastic controller still has the best performance, the fuzzy controller does better than that of Macfarlane and Belletrutti. This is simply because the fuzzy controller has a limited range of control outputs and has discretised inputs.

6. Conclusions and further work

These results highlight the two main problems in fuzzy controller design. Firstly, the derivation of the rules in the control algorithm and secondly, the implementation of these rules in a non-fuzzy environment such as a digital computer. Whilst in no way giving an answer to these problems, the results do show the importance of a careful choice of implementation parameters.

The results are encouraging, however, and work in the future will consider the effect of other implementation parameters such as the number of discretisation levels in the support sets and the choice of single-valued control from the output control

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set.

On a more general front, there is clearly a need for a coherent theory of control for fuzzy systems. Without it, each application can only be treated on an ad hoc basis and the ultimate usefulness of the fuzzy set approach will remain in doubt.

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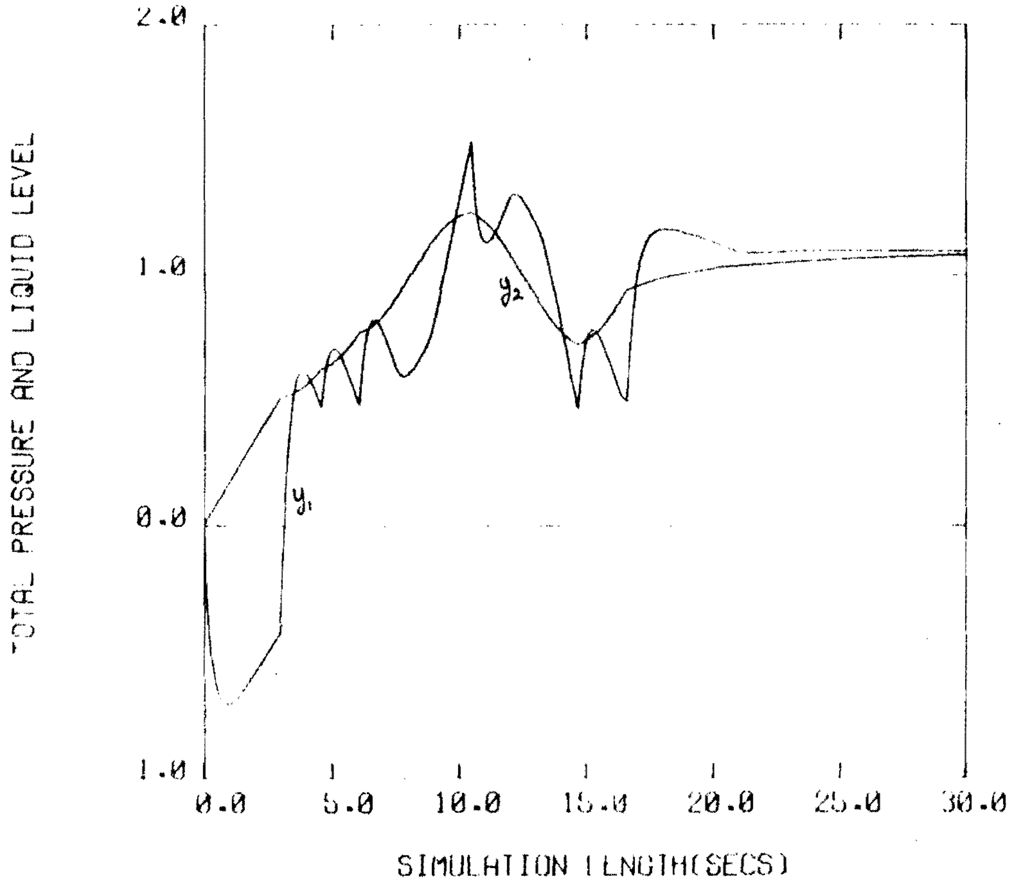


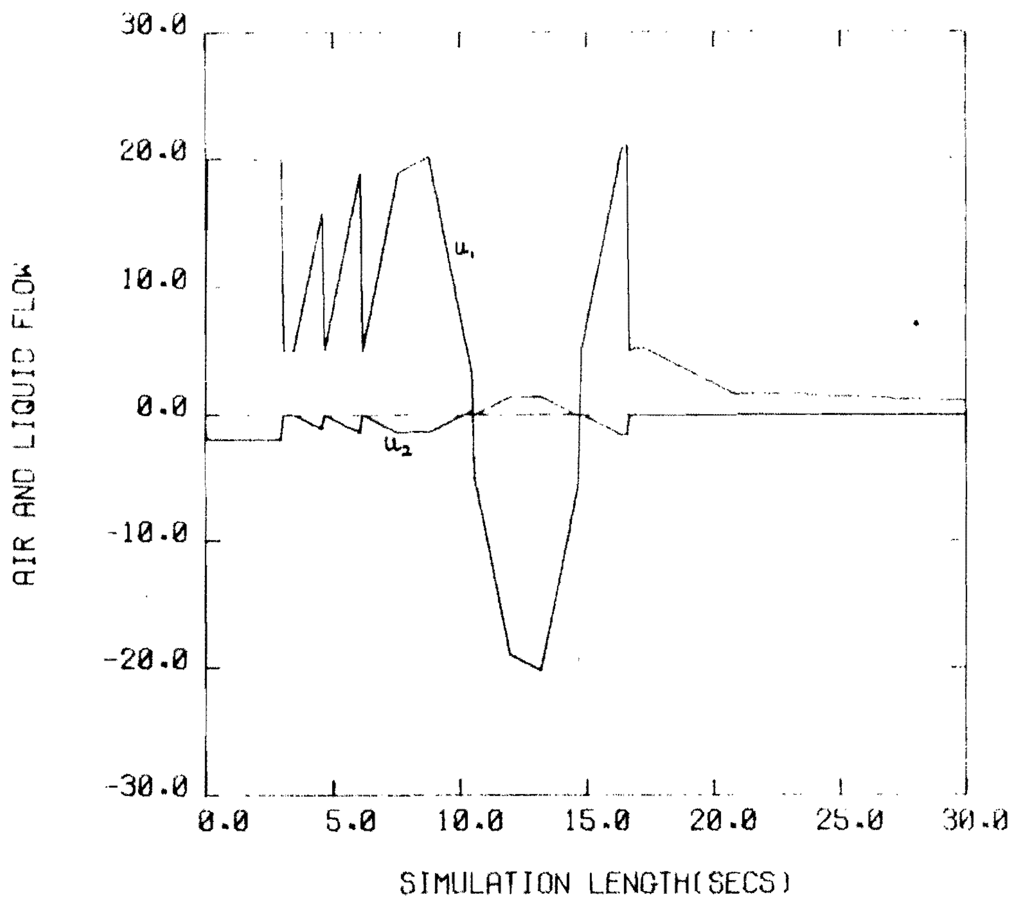
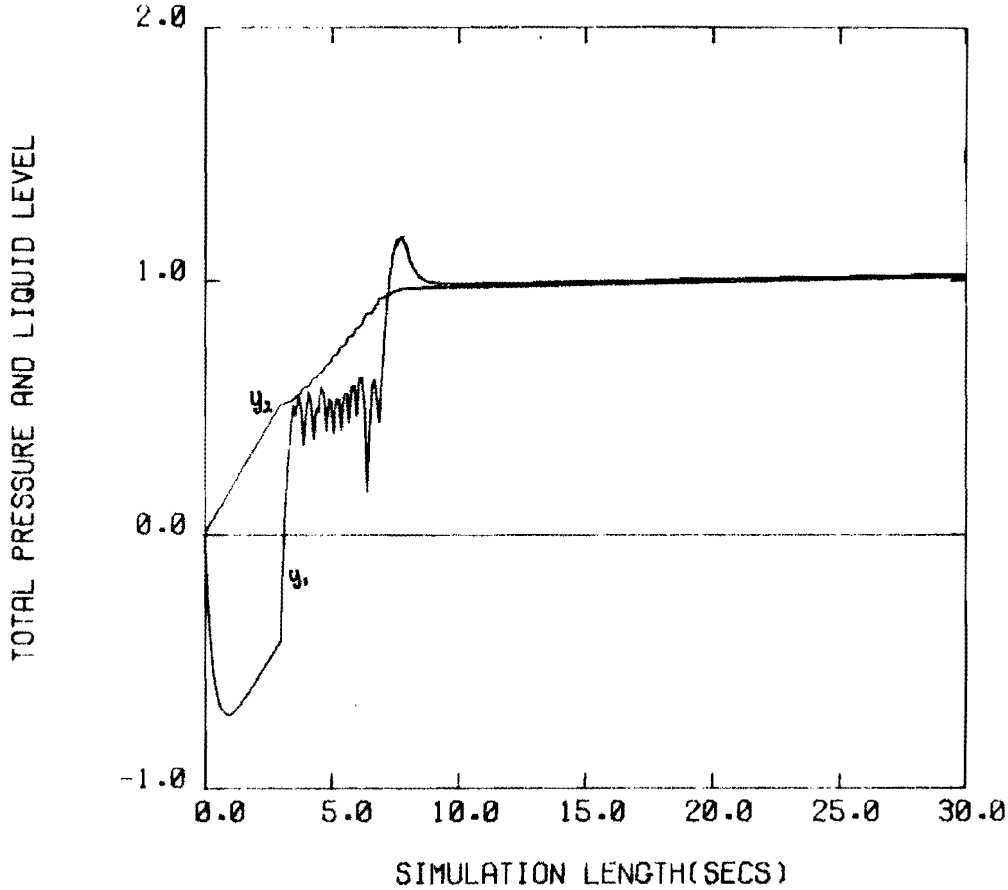


FIGURE 2.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y_1
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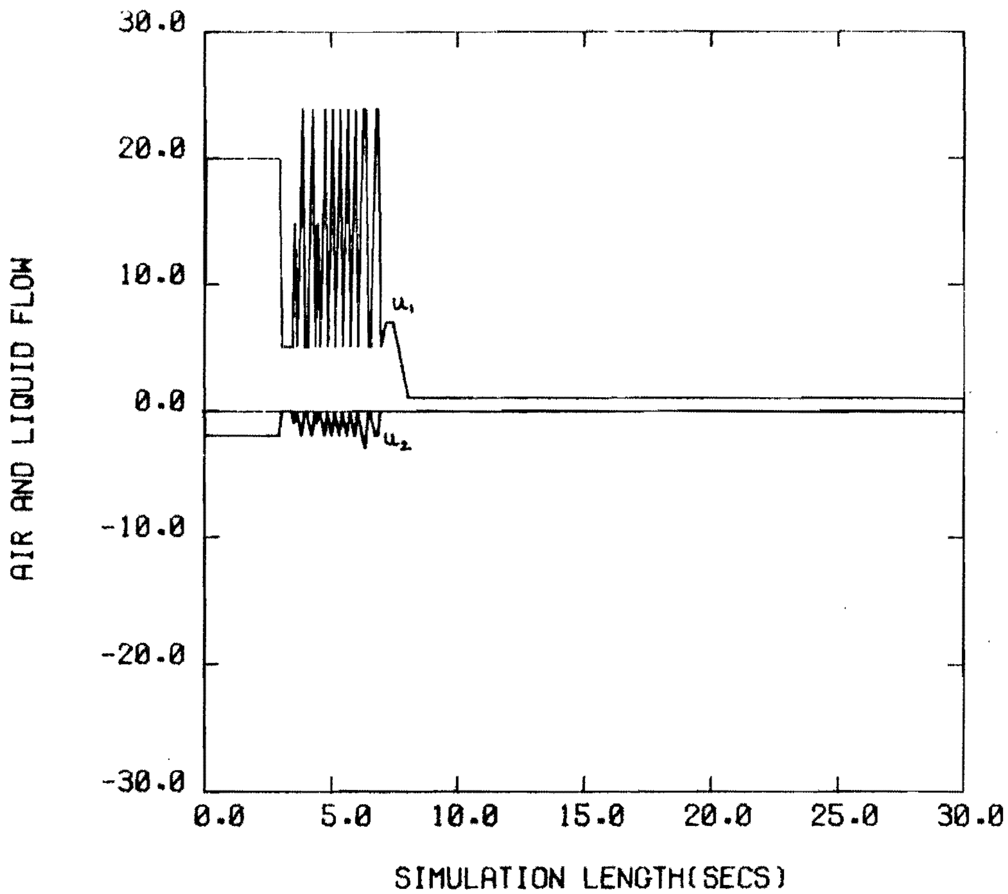
10.0
15.0
20.0
25.0
30.0
SIMULATION LENGTH(SECS)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-30.0
AIR AND LIQUID FLOW
u_1
u_2
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5.0
10.0
15.0
20.0
25.0
30.0
SIMULATION LENGTH(SECS)

Figure 3.

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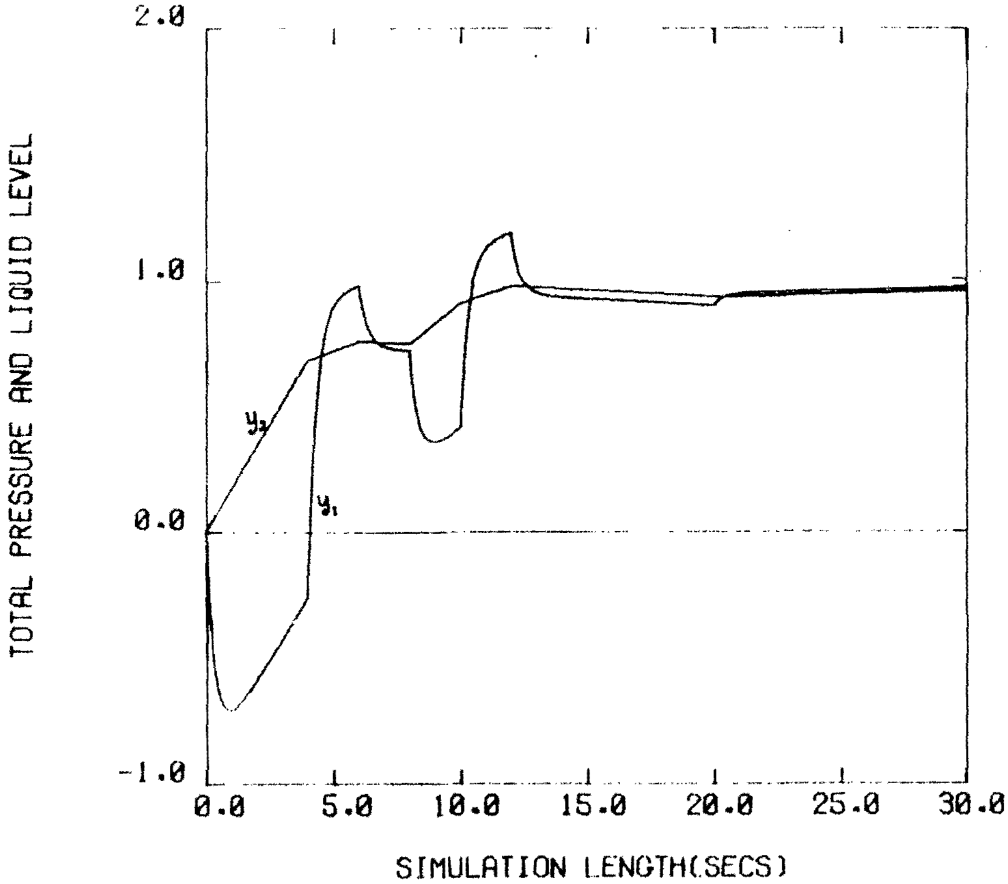


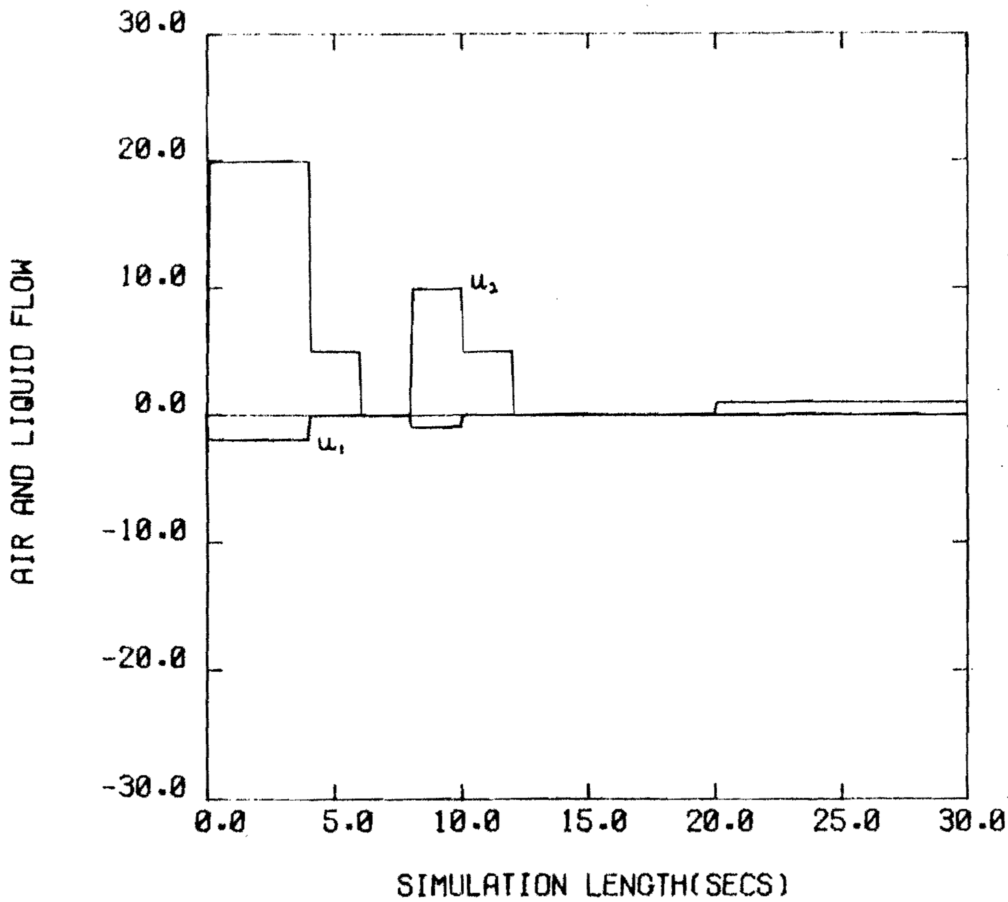
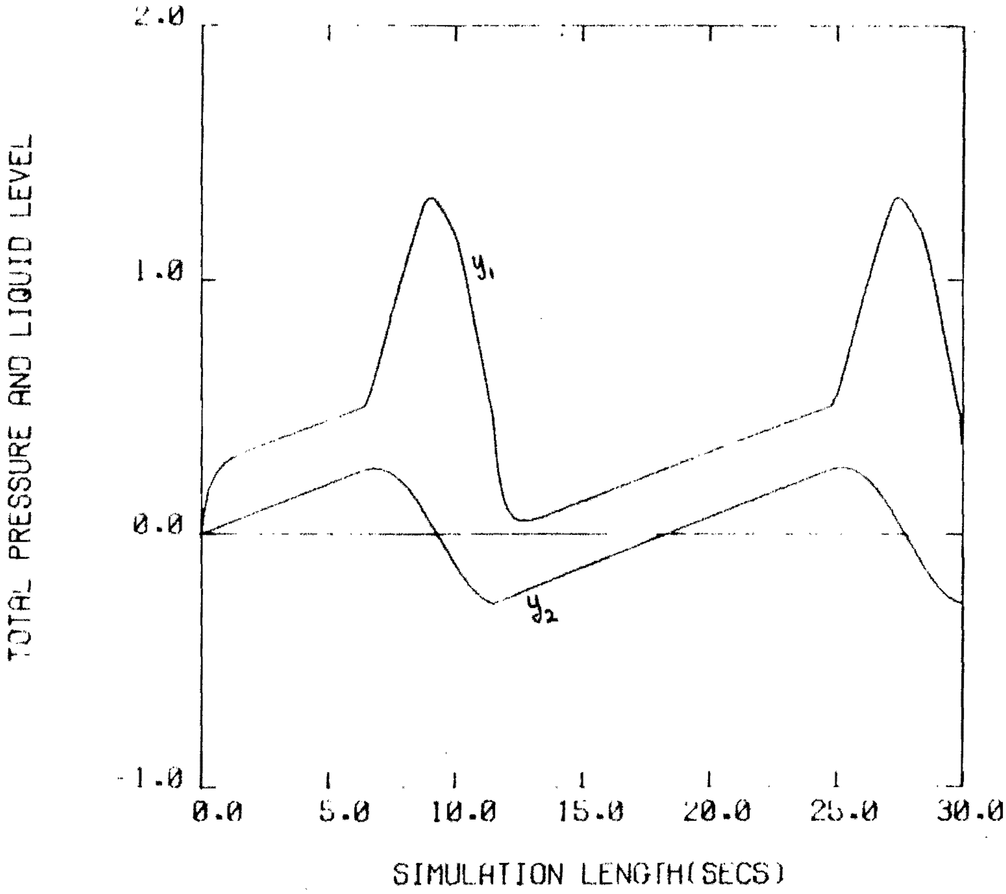


Figure 4.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TOTAL PRESSURE AND LIQUID LEVEL
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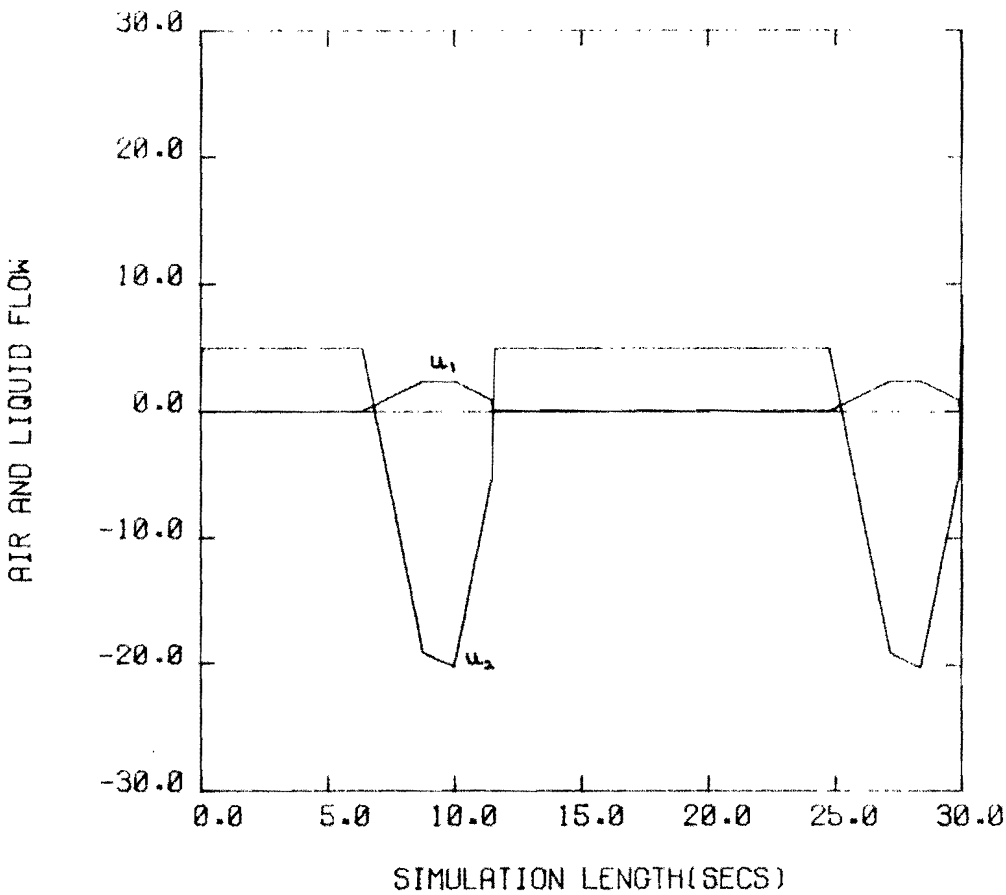
10.0
15.0
20.0
25.0
30.0
SIMULATION LENGTH(SECS)
y1
y2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20.0
10.0
0.0
-10.0
-20.0
-30.0
AIR AND LIQUID FLOW
0.0
5.0
10.0
15.0
20.0
25.0
30.0
SIMULATION LENGTH(SECS)
u1
u2

FIGURE 5.

5.19 /286

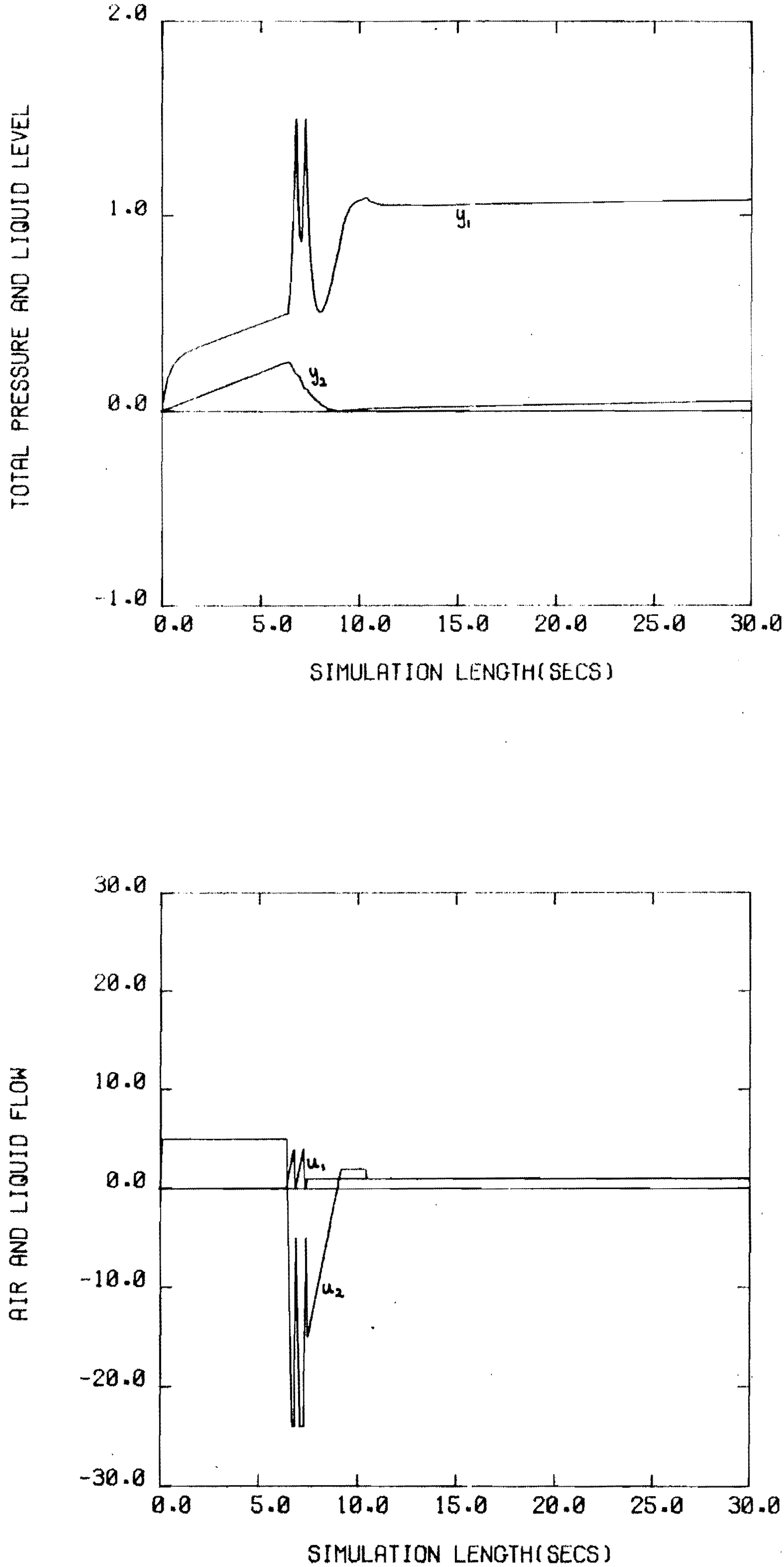
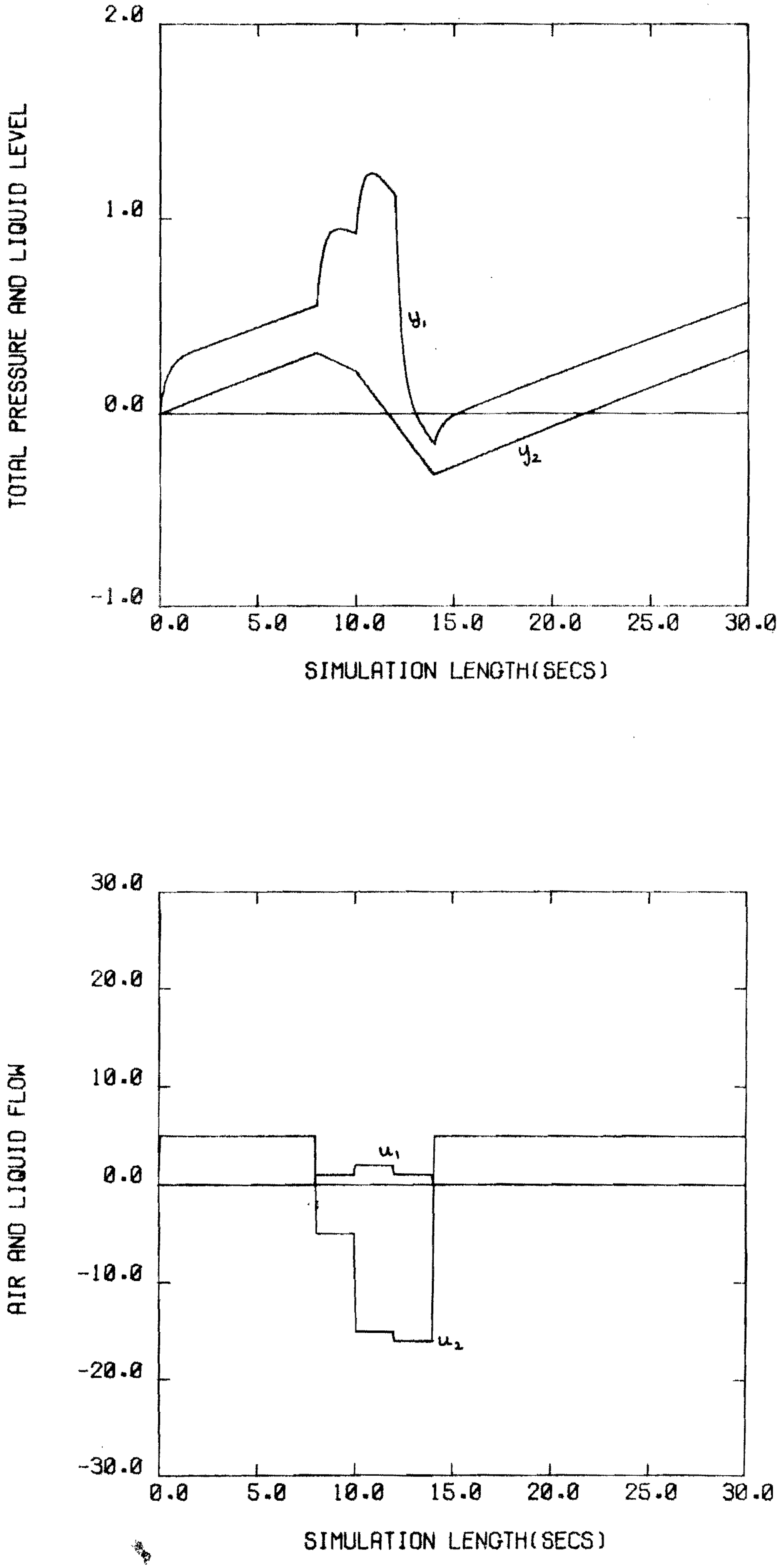


FIGURE 6.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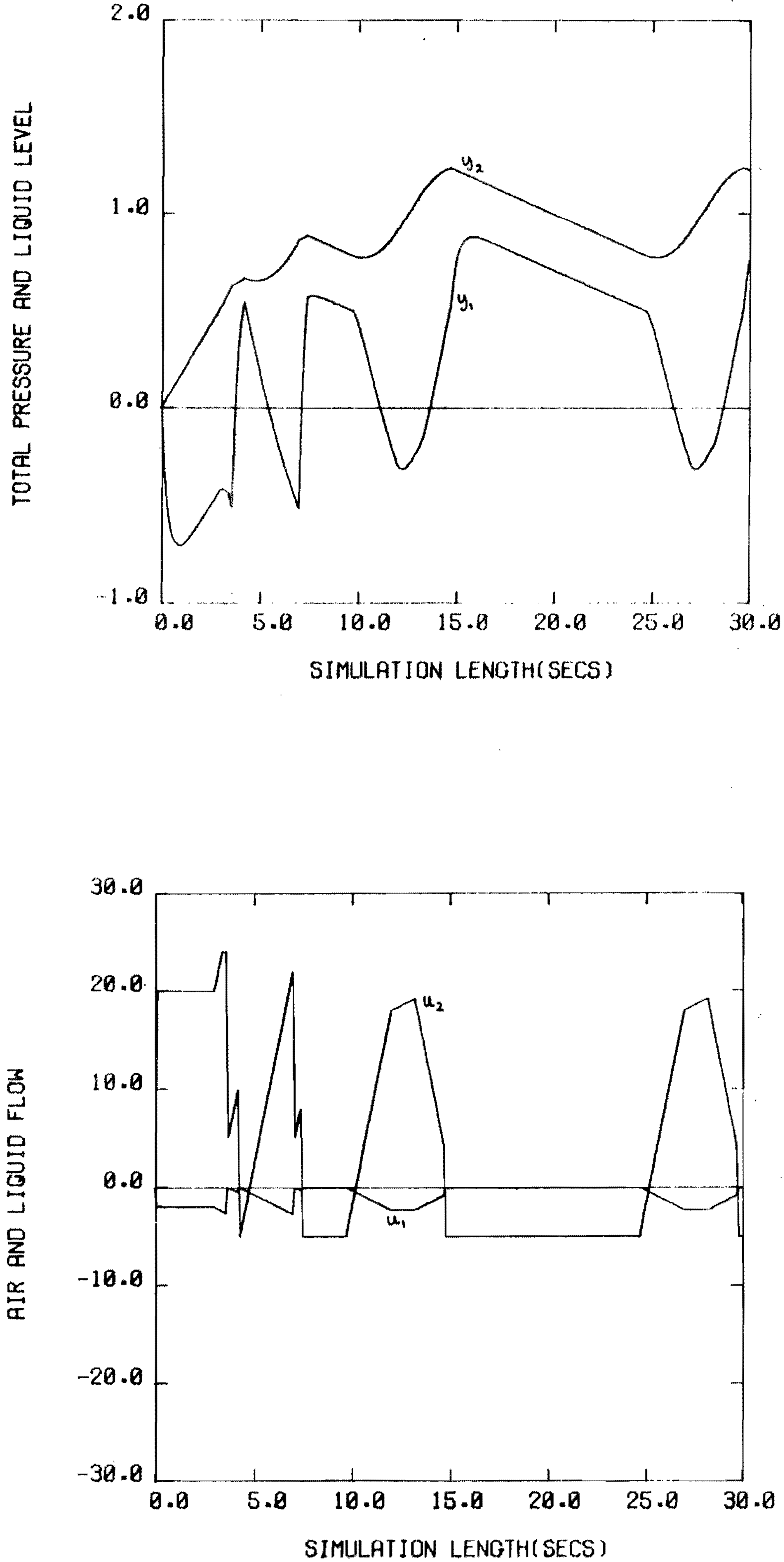
-1.0
0.0
5.0
10.0
15.0
20.0
25.0
30.0
SIMULATION LENGTH(SECS)
y_1
y_2
AIR AND LIQUID FLOW
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
0.0
5.0
10.0
15.0
20.0
25.0
30.0
SIMULATION LENGTH(SECS)
u_1
u_2

Figure 7.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-1.0
TOTAL PRESSURE AND LIQUID LEVEL
y2
y1
0.0
5.0
10.0
15.0
20.0
25.0
30.0
SIMULATION LENGTH(SECS)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
AIR AND LIQUID FLOW
u2
u1
0.0
5.0
10.0
15.0
20.0
25.0
30.0
SIMULATION LENGTH(SECS)

FIGURE 8.

S. 19/289

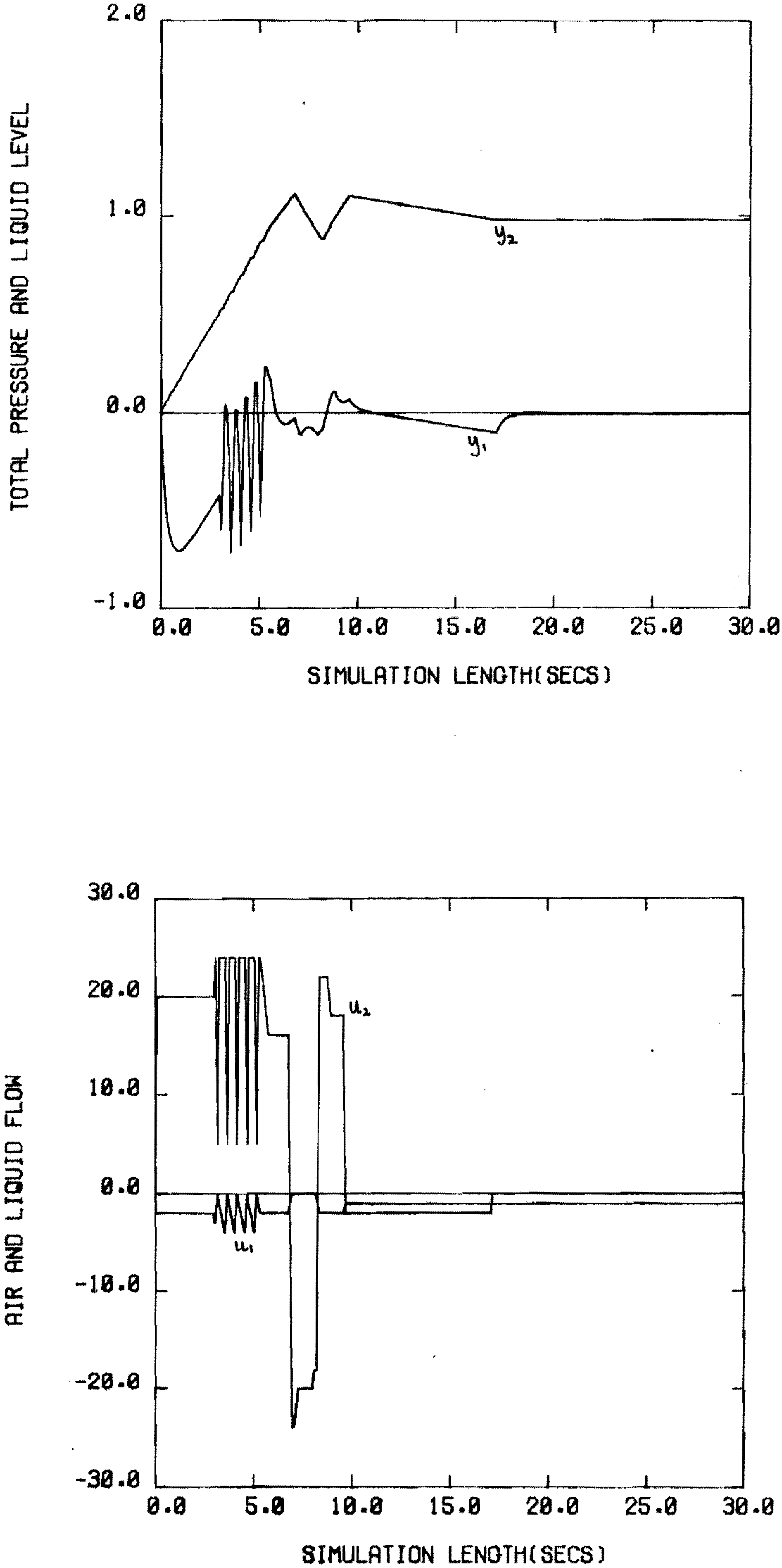

FIGURE 9.

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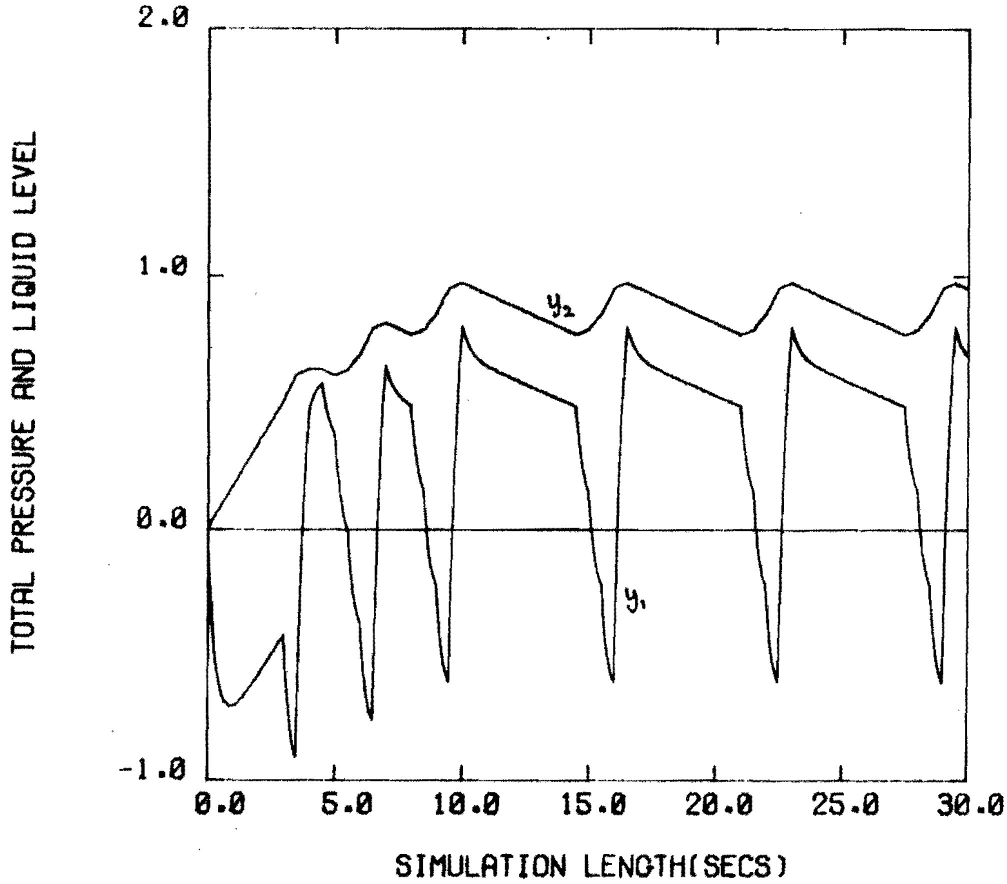


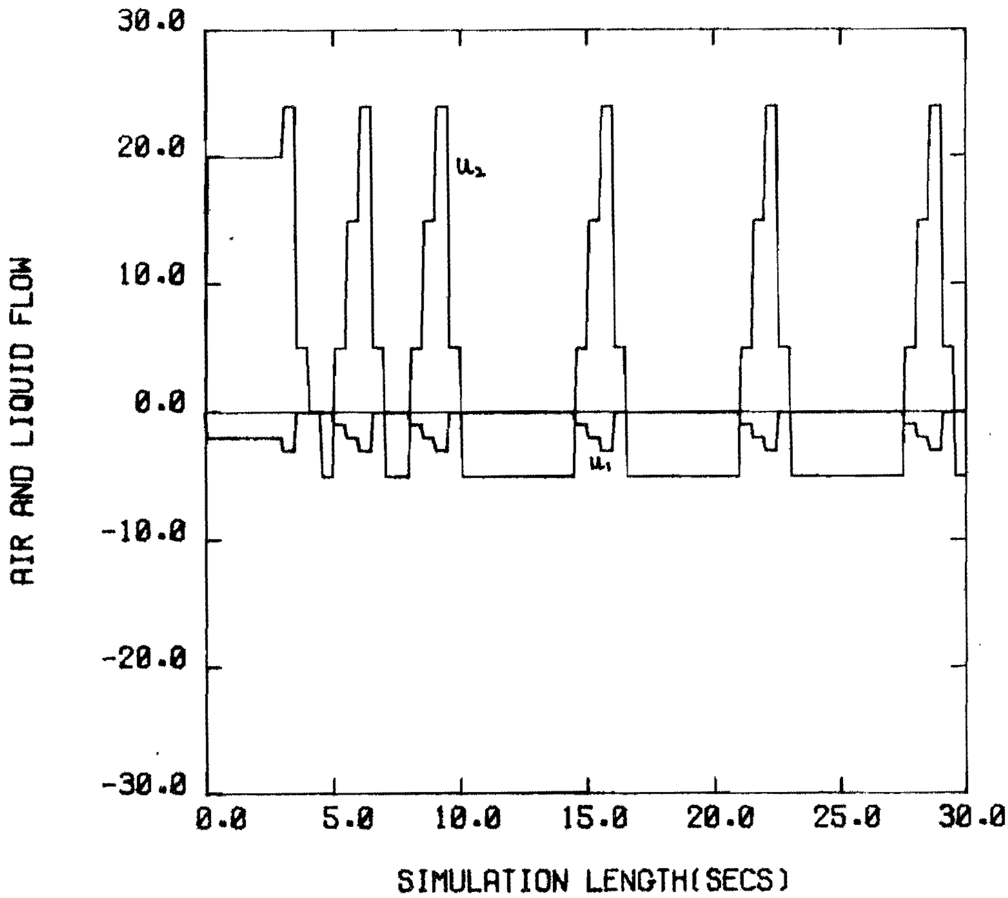
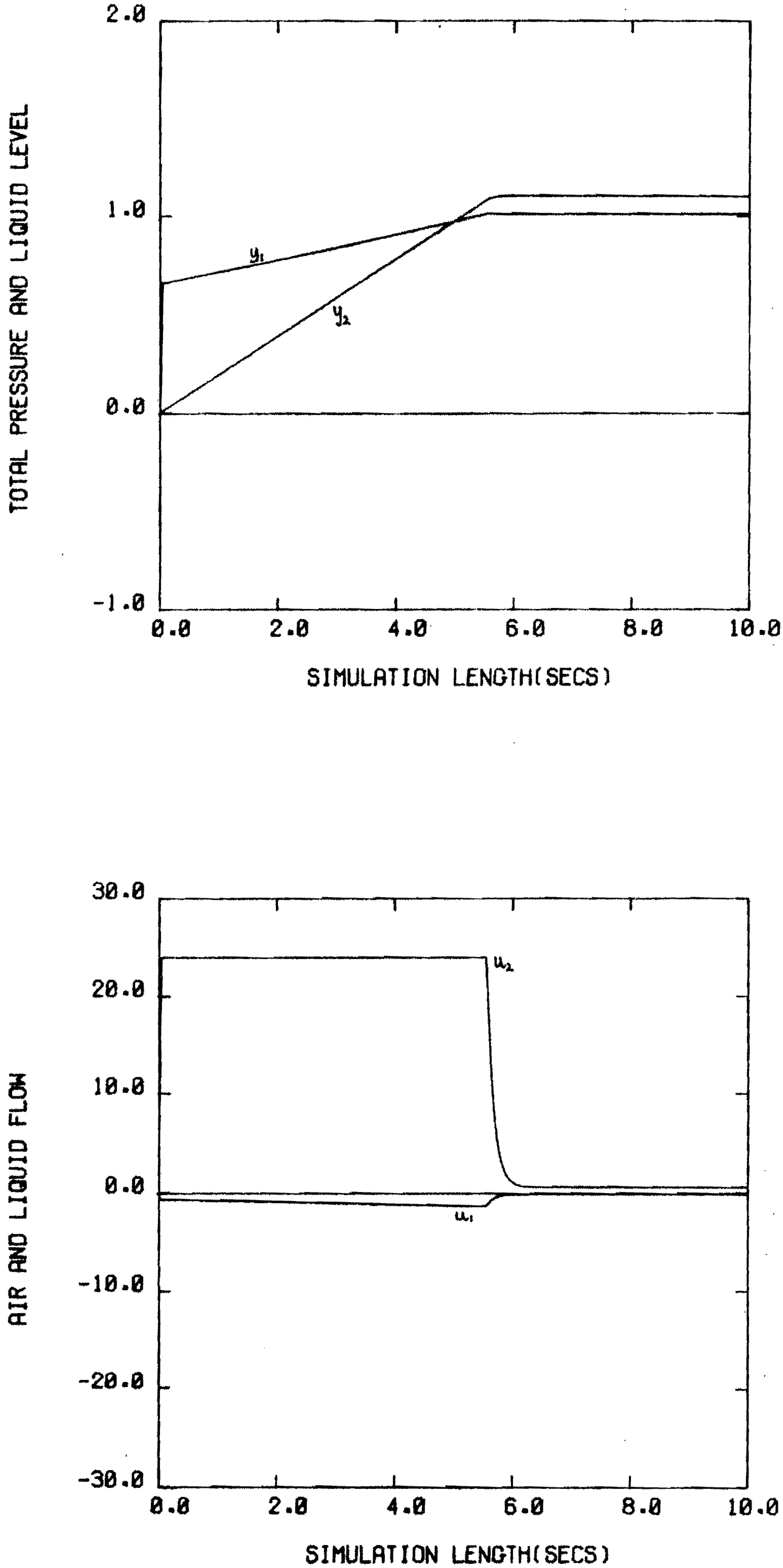


FIGURE 10.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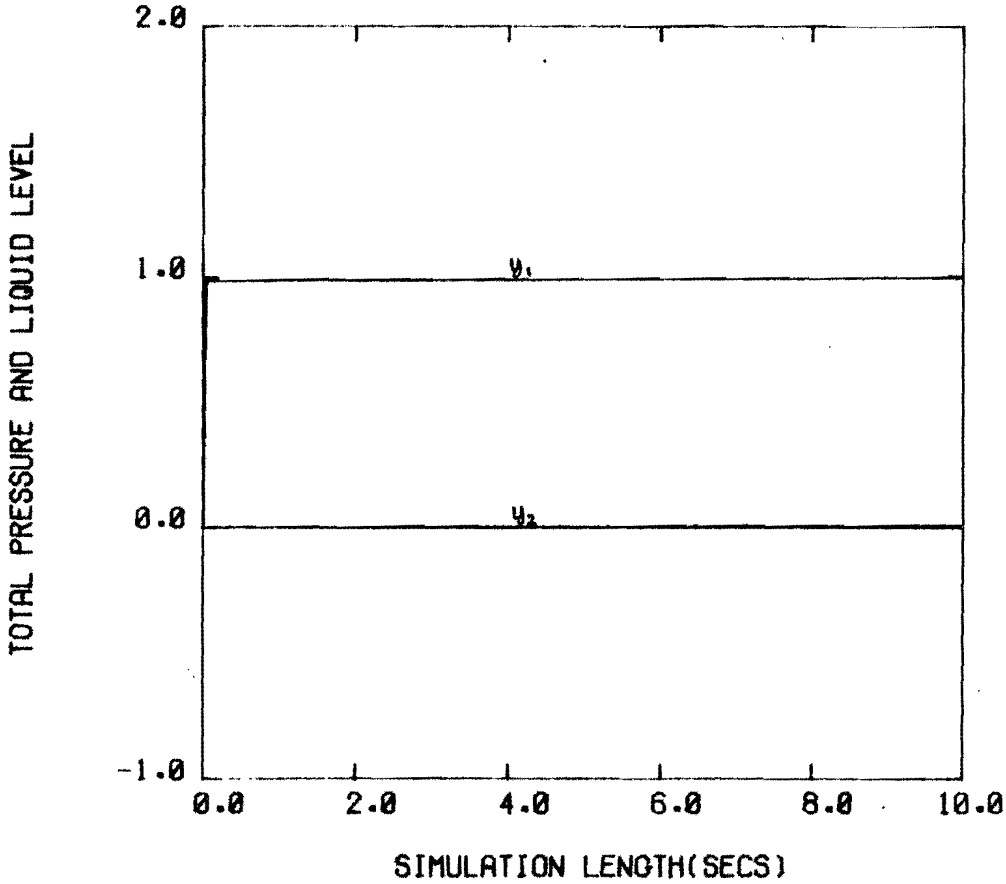
-1.0
TOTAL PRESSURE AND LIQUID LEVEL
y_1
y_2
0.0
2.0
4.0
6.0
8.0
10.0
SIMULATION LENGTH(SECS)
30.0
20.0
10.0
0.0
-10.0
-20.0
-30.0
AIR AND LIQUID FLOW
u_2
u_1
0.0
2.0
4.0
6.0
8.0
10.0
SIMULATION LENGTH(SECS)

FIGURE 11.



TOTAL PRESSURE AND LIQUID LEVEL
2.0
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0.0
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y_1
y_2
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2.0
4.0
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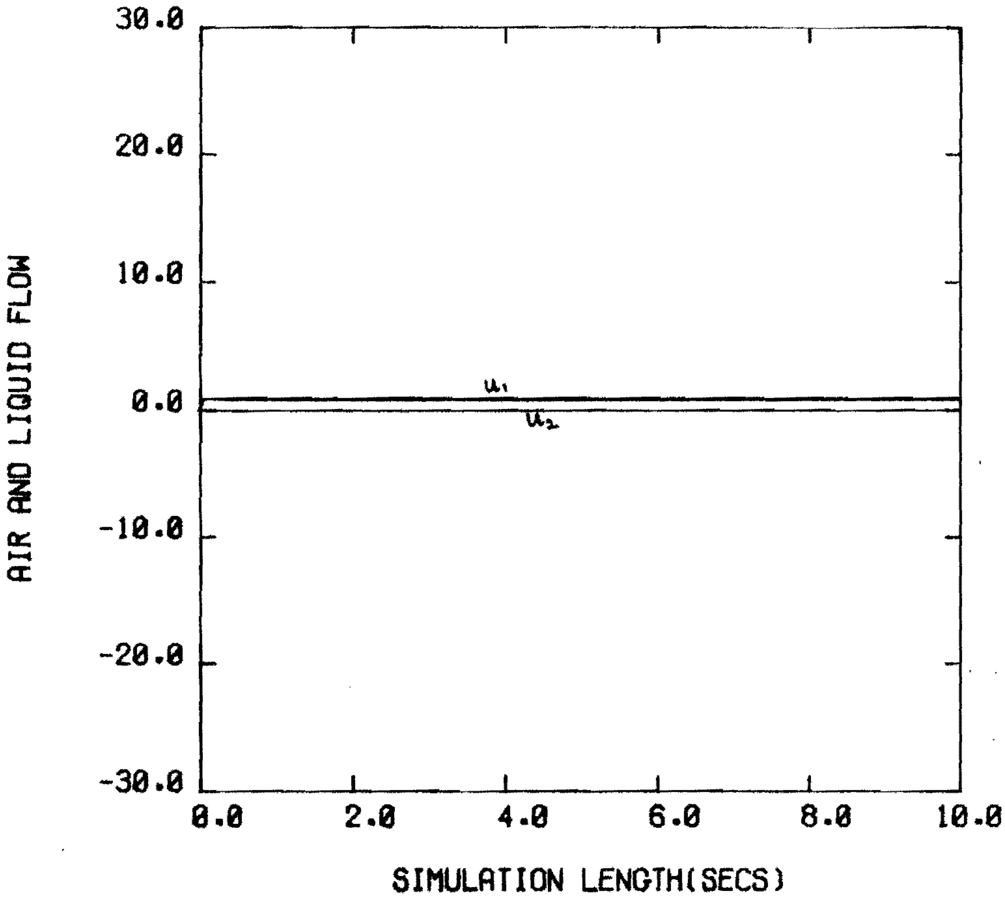
8.0
10.0
SIMULATION LENGTH(SECS)



AIR AND LIQUID FLOW
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-10.0
-20.0
-30.0
u_1
u_2
0.0
2.0
4.0
6.0
8.0
10.0
SIMULATION LENGTH(SECS)

FIGURE 12.

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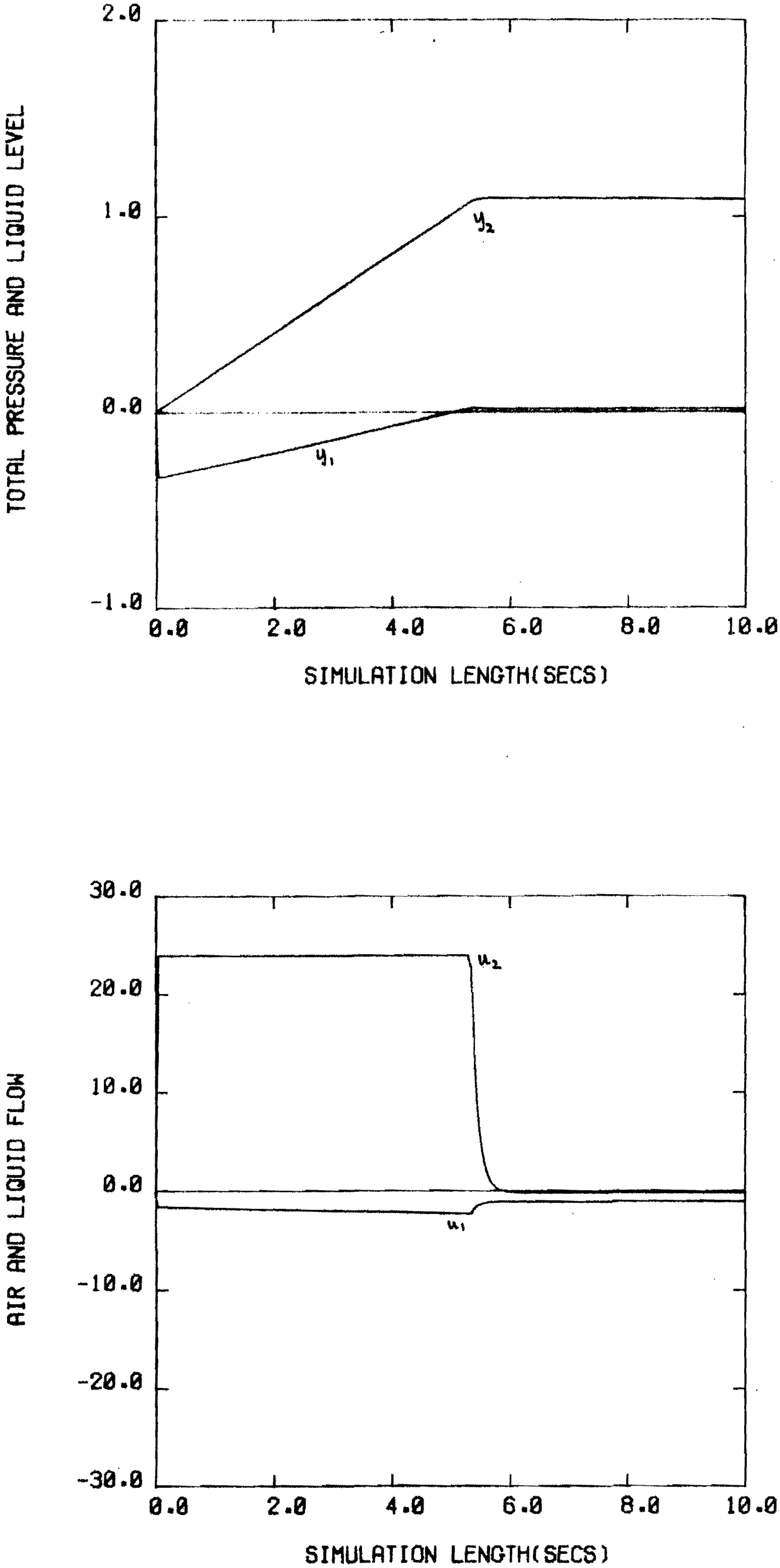
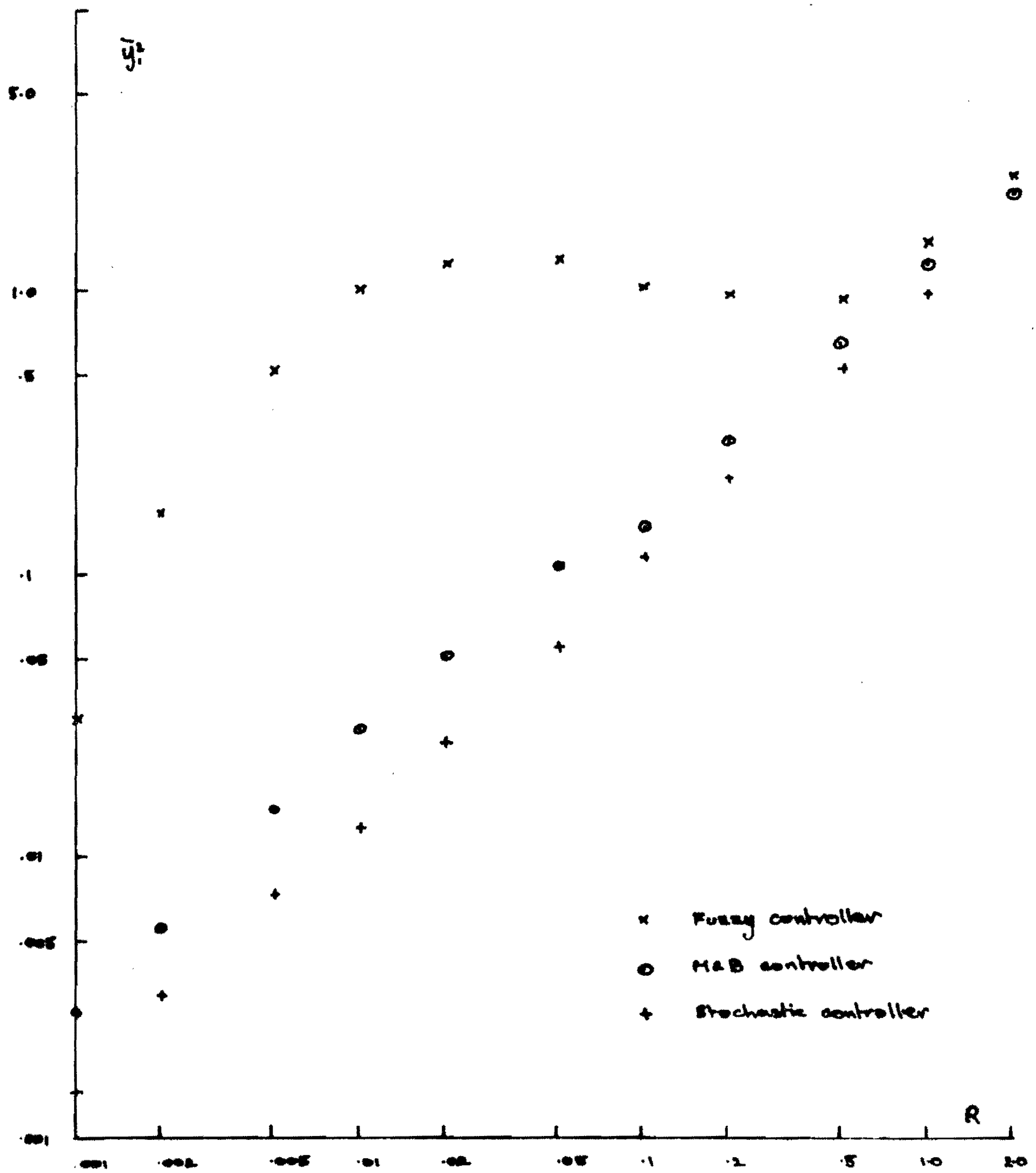
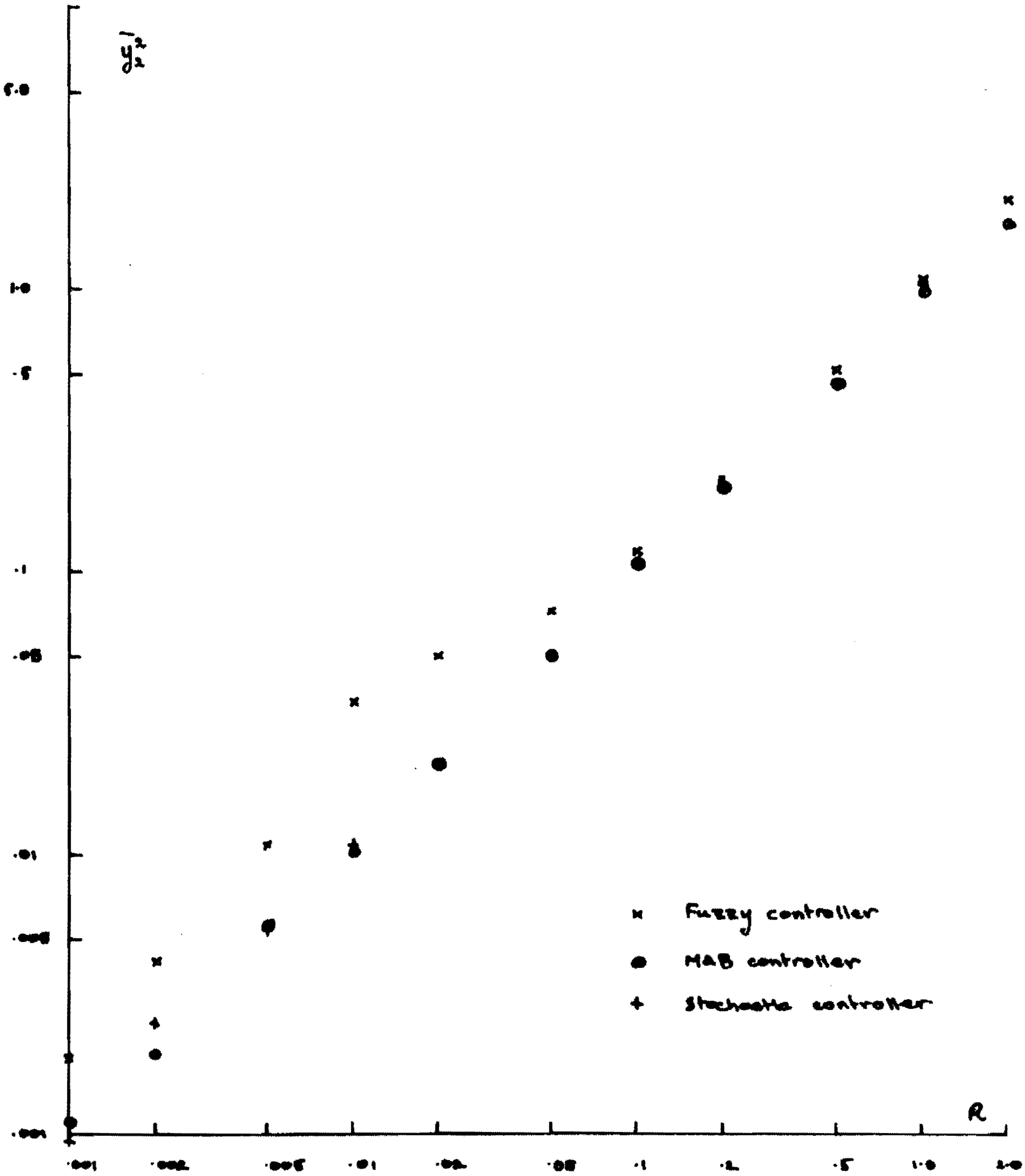


FIGURE 13.



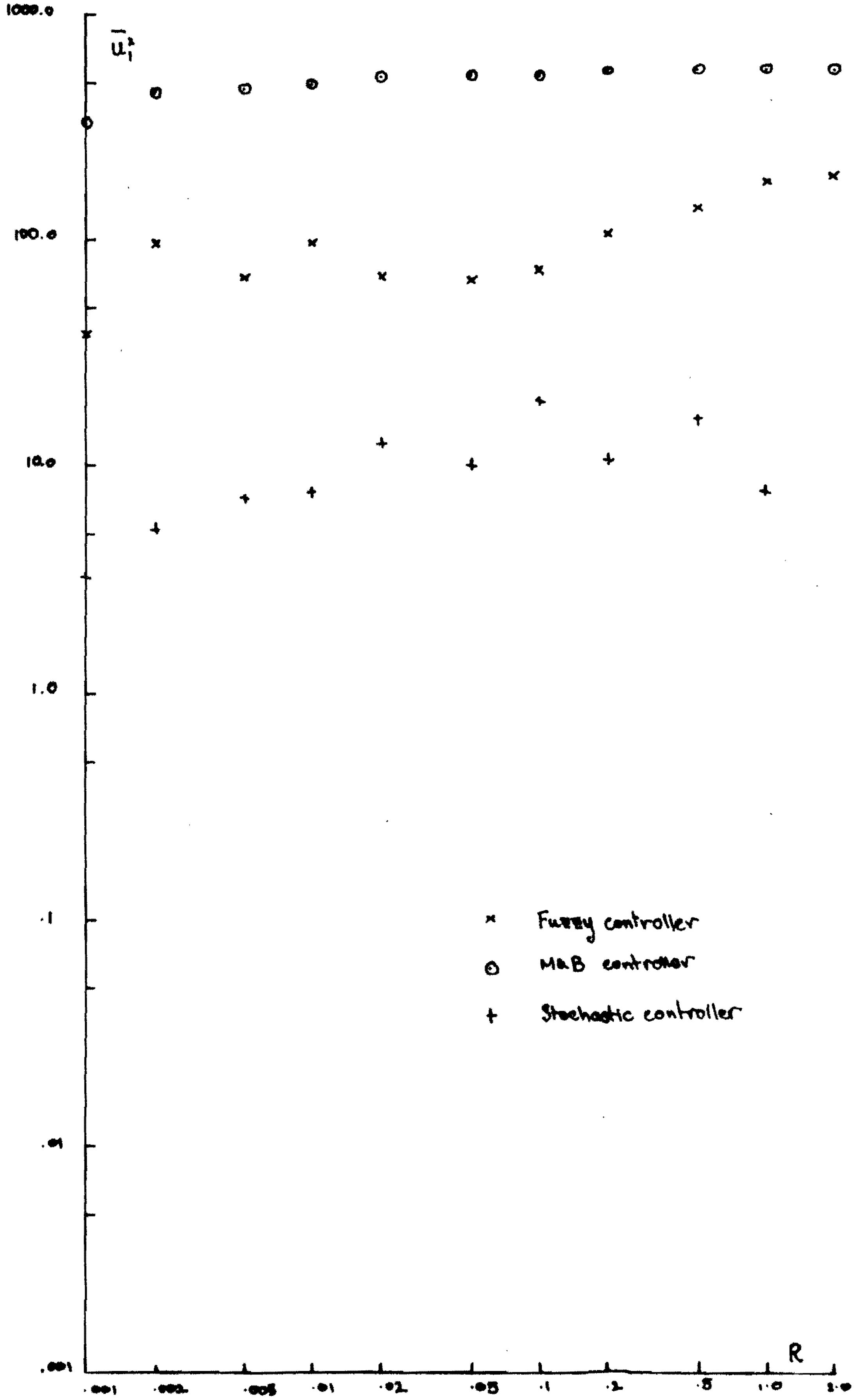
GRAPH SHOWING MEAN SQUARE OUTPUT AS A FUNCTION OF OBSERVATION NOISE VARIANCE

FIGURE 14.



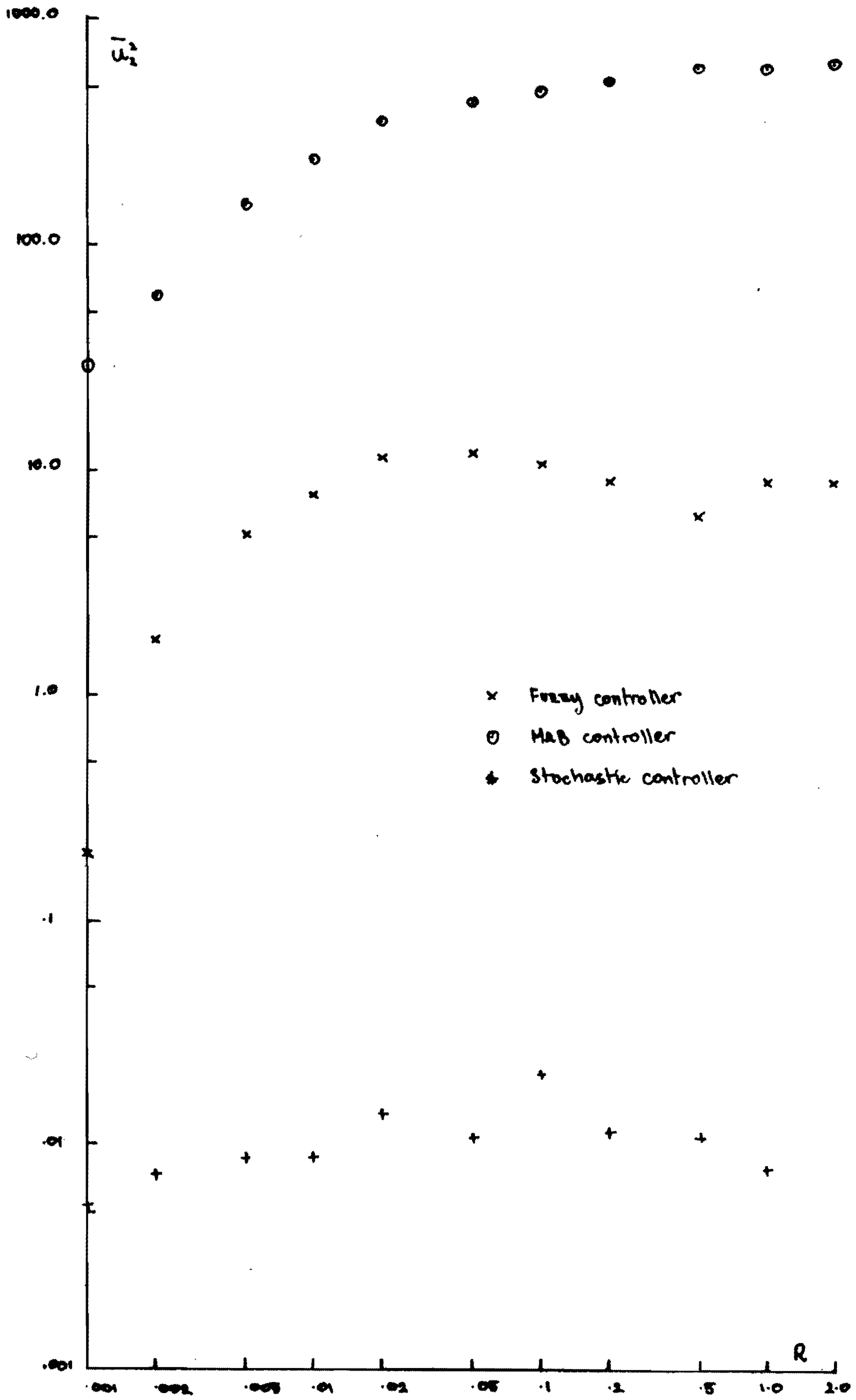
GRAPH SHOWING MEAN SQUARE OUTPUT AS A FUNCTION OF OBSERVATION NOISE VARIANCE

FIGURE 15.



GRAPH SHOWING MEAN SQUARE CONTROL AS A FUNCTION OF OBSERVATION NOISE VARIANCE

FIGURE 16.



GRAPH SHOWING MEAN SQUARE CONTROL AS A FUNCTION OF OBSERVATION NOISE VARIANCE

FIGURE 17.