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THE LOGIC OF AUTOMATA

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Automata are the prime example of general systems over discrete spaces, and yet the theory of automata is fragmentary and it is not clear what makes a general structure an *automaton*. This paper investigates the logical foundations of automata relating it to the semantics of our notions of *uncertainty, state* and *state-determined*. A single framework is established for the conventional spectrum of automata: *deterministic, probabilistic, fuzzy*, and *non-deterministic*, which shows this set to be, in some sense, *complete*. Counter-examples are then developed to show that this spectrum alone is *inadequate* to describe the behaviour of certain forms of uncertain system. Finally a general formulation is developed based on the fundamental semantics of our notion of a state that shows that the logical structure of an automaton must be at least a *positive ordered semiring*. The role of *probability logic*, its relationship to *fuzzy logic*, the roles of *topological* models of automata, and the symmetry between inputs and outputs in *hyperstate/hyperinput-determined* systems are also discussed.

INDEX TERMS Automata, state, logic, probability, fuzzy, deterministic, non-deterministic, modal, multi-valued, semiring.

1 INTRODUCTION

There is a danger in all general systems theories that the generality may be carried too far. That is the possibilities encompassed by the formalism may go beyond those required by the semantics of any application or, worse, the "general" case may include instances that have apparent applications but which actually conflict with the assumed semantics. In this paper we are concerned with the semantics of discrete, state-determined systems, or *automata*, and with the most general formalism that encompasses all cases of interest and yet adheres strictly to the semantics of our notions of "state" and "automaton".

Automata theory as a subject area grew naturally out of the work of Turing[1] in the thirties on the mathematics of computation in which the notion of an abstract, state-determined machine played a major role. This concept was immediately attractive not only in its obvious role as a foundation for the design of relay switching circuits and the nascent digital computer, but also as a model of biological phenomena in neural networks (e.g. the work of McCulloch and Pitts[2]). The wide ranging interests of von Neumann[3] encompassing, and making major contributions, to both computers and biology firmly established this dual role of automata theory in the early forties quite independently of any speculations about the relationship between the computer and the brain.

Thus automata theory developed as a general systems tool from the beginning. However, it is interesting to note that in the late forties Wiener[4] in proposing the integrative viewpoint of *cybernetics*, and Bertalanffy[5] in proposing the even wider ranging viewpoint of *general systems theory*, exemplify their approaches with the differential equations of continuous systems rather than the discrete space formulations of automata theory. It was left to the brilliant expositions of Ross Ashby[6, 7] in the early fifties to demonstrate the major role of automata theory, complementary to that of continuous systems theory, in such general approaches to natural and artificial systems.

The joint origins of automata theory in biology and computer engineering have been succinctly reviewed recently by Burks.[8] In a related survey Arbib[9] criticizes the applicability of current automata theory and suggests that many new developments and extensions are required. This criticism will be echoed by those who have recognized the concepts of automata theory as relevant to their own disciplines but have been disappointed in the dearth of applicable results.

The convictions, on the one hand, that the basic concepts of automata theory are relevant but, on the other, that the present developments are not sufficiently fruitful have prompted several workers' to investigate new automaton structures, e.g. Arbib's restriction of state transitions to represent generalized continuity in *tolerance automata*[10] and Zadeh's generalization of state transitions to represent non-probabilistic imprecision in *fuzzy automata*.[11] Through the very diversity of interests involved automata theory has grown up piecemeal with a variety of automaton structures and semantic interpretations. The continuing intermittent addition of new structures reinforces the impression that not just the development of the subject but perhaps also its foundations are, in some sense, incomplete.

This paper was motivated by our own experience in applying algebraic system theory to problems of system identification, stability and control, where we have found it necessary to define automaton structures that do not fit the conventional spectrum of deterministic, stochastic, fuzzy and non-deterministic automata. These new structures initially appeared to be representable as automata over modal logics rather than Boolean algebra. However, the need soon became apparent for mixed logics involving continuous probability intervals as well as discrete modalities, and the variety of possibilities led us to look for some more general approach.

It has been shown by Santos and Wee[12] that the main spectrum of deterministic, stochastic, fuzzy and non-deterministic automata can be fitted into a single formalism, but this is descriptive rather than axiomatic. It leaves open many questions: whether further automaton structures can be invented ad infinitum; what is the most general formulation; and so on. The search for generality is itself dubious unless backed by definite practical requirements expressed as semantic constraints. In this paper we take three distinct approaches to the problem of establishing the most general structure possible for an automaton: analysing first a sense in which the conventional spectrum of automata is already *complete*; secondly arguing from practical application requirements that this spectrum is *inadequate*; and thirdly, reversing the direction of increasing generality, to show by foundational arguments that certain quite powerful structural constraints are *necessary* to an acceptable concept of an automaton, i.e. that arbitrary algebraic structures formally similar to automata do not necessarily possess viable semantics.

2. CONVENTIONAL AUTOMATON STRUCTURES

2.1. The Generalization to Hyperstates and Hyperinputs

The key concepts in automata theory are clearly those of a *state* and the behaviour of a system being *state-determined*. Both the role of these concepts in modern system theory and their formal status have been lucidly analysed by Zadeh[13,14]. Given that an automaton is a discrete-time, discrete-state-space, state-determined machine, at first sight there appears little scope for generalization. The entire structure is well-defined and may be presented as a function mapping the current state and input into the next state (discussion of the role of the state-dependent or state/input-dependent *output* will be deferred to section 4).

However, neither the actual current state of an automaton nor its current input are necessarily well-defined. For example, we may know only the *probability distribution* of possible current states, or of possible current inputs. In either case the next state of the automaton will not necessarily be a single state but will probably also be known only as a distribution. It is a convenient generalization of the concept of an automaton to consider transitions not just between states but between such state distributions, regarding distributions over states and inputs as generalized "states" and "inputs", respectively (the terminology of *hyperstates* and *hyperinputs* is convenient in making this generalization).

It is this extension of the concept of state/input-determined to what might be called hyperstate/hyperinput-determined systems that we shall analyse. Note that the basic concept of a state still requires that the automaton be regarded as being in only one state at a time, although the actual current state may be uncertain. Those hyperstates that correspond to no uncertainty, to the automaton actually being in a single state, will be called *sharp*.

The conventional generalizations of deterministic automaton to probabilistic, fuzzy, and non-deterministic automata are examples of allowing certain forms of hyperstate. In the following sub-sections we shall first consider a common notation for each of these forms and then analyse in detail the relationships between them.

2.2 *V-sets and Normalization*

The conventional forms of hyperstate can all be represented as mappings from the set of states, S, to a truth-set, V, $\delta: S \to V$. We shall adopt in the paper a notation for mappings of states that avoids parentheses and subscripts since both often obscure the essential simplicity of automaton operations. δ will be treated as a unary operator binding on the right so that we may write δs for the image in V of $s \in S$. Goguen[15] calls a mapping such as δ a *V-set* with S as *carrier*.

For the purposes of describing automata states we also require a *normalization* condition expressing that the automaton is actually in one and only one state. We shall later take V to be a semiring with binary operations, $\oplus$ (we do not use "+" because it can be confused with arithmetic +) which is associative and commutative, and $\circ$ which is associative, often commutative, and distributes over $\oplus$. Both $\oplus$ and $\circ$ will be regarded as infix operators mapping $V \times V \to V$ such that δ takes precedence over $\circ$ which itself takes precedence over $\oplus$. It is convenient to express the normalization condition in terms of the formal expression, $\bigoplus_{s \in S} \delta s$, meaning the result of operating over the entire co-domain of δ in the truth-set with $\oplus$ (i.e. "summation" if $\oplus$ is actually + —we assume such an operation is well-defined if S should be infinite). By suitable choice of V and $\oplus$ we will show that the normalization for all four cases to be considered may be taken to be:

$$\bigoplus_{s \in S} \delta s = 1 \tag{1}$$

where 1 is a "zero" in V for $\circ$.

As a further aid to brevity in notation we shall adopt a convention, similar to that of the tensor calculus, that summation over repeated dummy variables is implicit. In most cases of interest such a repetition naturally arises—we can introduce it artificially into eq. 1 by taking λ to be a mapping from S to 1 in V, i.e. $\lambda s = 1$ for all $s \in S$. Then eq. 1 may be written:

$$\lambda s \circ \delta s = 1 \tag{2}$$

with implicit "summation".

2.3 *Deterministic States*

These express the conditions that arise when a system's behaviour is completely defined and determinate. The automaton representing it is always in a well-defined, "sharp", state. We can express this: for each state, it is true or false that the automaton is in the state *and* the automaton is in precisely one state. A suitable truth set is binary, $V \equiv \{0, 1\}$, with $\oplus$ being arithmetic +, and the normalization as in Eq. (1). This necessitates only one state being mapped onto 1, and hence we could express the normalization as, "the inverse image of 1 under δ contains just one element."

One note in passing that we shall discuss further in section 2.9 is that it is essential to take $\oplus$ for a deterministic automaton to be arithmetic + and not logical **OR** (perhaps a more obvious choice). There is no equivalent to the normalization of Eq. (1) if $\oplus$ is taken as **OR** and hence no convenient way of expressing that only one state is possible.

2.4 *Probabilistic States*

These express the conditions that arise when a system's behaviour is a Markov process whose behaviour is constrained by well-defined probabilities. The *probability* of the automaton representing it being in a particular state is then always well-defined. That is, for each state, the probability that the automaton is in the state is defined and the automaton is in precisely one state (the probabilities over all states sum to one and the conditional probabilities of the automaton being in one state given that it is in another are all zero). A suitable truth set is a closed interval of reals, $V \equiv [0, 1]$ say, with $\oplus$ being arithmetic +, and the normalization as in Eq. (1).

2.5 *Fuzzy States*

Zadeh's concepts of fuzzy logic[16] and fuzzy automata[11] represent an attempt to provide a formal basis for a calculus of approximate reasoning.[17,18,19] Formally fuzzy logics in their basic forms are closely related to the various classical multi-valued logics[20] of Lukasiewicz, Dienes, Gödel, etc. However Zadeh has contributed new and practically interpretable semantics that makes the application of these logics attractive in systems engineering, for example, in pattern recognition,[21] taxonomic clustering,[22] process control,[23] robot planning,[24] and many other applications.[25]

Thus one view of the concept of a "fuzzy" state is that it expresses the form of hyperstate that may arise when a system's behaviour is being described by a process of approximate reasoning. Other interpretations are possible and we shall adopt a formal viewpoint in this paper, being concerned only with the consistency of the notion of a fuzzy state with that of a state itself. Formally, for a fuzzy state the *degree of membership* of each particular state of the automaton to being the actual state is defined. If we take the usual fuzzy logic system with the truth set being the closed interval of reals and $\oplus$ being a **MAX** operator, then $V \equiv [0, 1]$ and $a \oplus b = \mathbf{MAX}(a, b)$.

2.5.1 Normalization of fuzzy states. The normalization of fuzzy state sets to express the condition that the automaton is actually in precisely one state requires special attention. The published semantics of fuzzy automata are unclear on this point. Wee and Fu[26] suggest that if a state has a degree of membership of unity then the automaton is definitely in the associated state. However, the converse is not true and it is possible for the rules of fuzzy logic to generate a situation in which the degrees of membership of all states are zero except one which is *not* unity. This is so even if the total degree of membership is "normalized" as suggested in Ref. 26 in the same way as a stochastic automaton (arithmetic sum of degrees of membership being unity). It also leaves open the meaning of two distinct states each having a degree of membership function of unity—an important case since it corresponds to the classical non-deterministic automaton.

It seems better to place the emphasis on the degree of membership of a state being zero as implying that the automaton is *not* in the associated state. With the usual fuzzy logic definitions of V and $\oplus$ given in section 2.5, our normalization condition of Eq. (1) requires only that at least one state has a degree of membership of unity. This condition is consistent with the definition of $\oplus$ in fuzzy logic, whereas the proposed "normalization" of Ref. 26 introduces arithmetic $+$, an operator outside fuzzy logic. Neither normalization is consistent with a degree of membership of unity implying that the automaton is definitely in the associated state, and this requires an alternative definition.

A similar problem arises with non-deterministic automata and is clearly a semantic one to be resolved in actual applications. The formal normalization condition proposed here retains consistency between fuzzy automata and the others. We would propose the interpretation that a fuzzy automaton is definitely in a state if the truth values of all the *other* states are zero. The normalization of Eq. (1) then implies that the truth value of the remaining state is unity—the converse is not true.

2.6 Non-deterministic States

These might more positively be called "possibilistic" since they express the conditions that arise when a system's behaviour is such that only the *possibility* and *impossibility* of its being in a given state can be discriminated. That is, for each state either it is possible, or impossible, that the automaton is in that state and the automaton is in precisely one state (at least one state is possible, and if only one state is possible than the automaton is in that state). A suitable truth set is binary, $V \equiv \{0, 1\}$, with $\oplus$ being Boolean **OR** which also corresponds to the **MAX** operation over this truth set. The normalization of Eq. (1) implies that the inverse image of 1 under δ contains *at least one* element (as it also does for fuzzy states).

2.7 From States to Transitions

Having given appropriate forms for the hyperstates of deterministic, probabilistic, fuzzy, and non-deterministic automata, we shall next examine the forms of the state transitions in these cases, defining appropriate next-state-functions, NSF's. For the moment we shall consider only the NSF corresponding to a particular input, or hyperinput, to the automaton. A full definition will involve a family of such NSF's. In this section also we shall take it for granted that the nature of transitions can be expressed in terms of the same truth set as that for the states themselves. For example, a "stochastic automaton" is one with stochastic states and stochastic state transitions. We can express the NSF as a function, $\sigma: S \times S \rightarrow V$, mapping a pair of states into the truth set—which represents the truth value of the transition from one state to the other. Again, for notational convenience, σ will be regarded as an infix operator taking precedence over $\oplus$, $\circ$ and δ, such that, $p\sigma s$, is the value in V of the transition from state p to state s—note that the operation σ is not commutative in general.

The normalization of σ for the cases considered may be expressed as:

$$p\sigma s \odot \lambda s = 1 \tag{3}$$

for all $p \in S$ (with implicit summation over all $s \in S$). And, if δ' is the new V-set function determining the hyperstate after a transition, we have:

$$\delta' s = \delta p \odot p\sigma s \tag{4}$$

As a matter of notation we can write this as an equation for δ':

$$\delta' = \delta p \odot p\sigma \tag{5}$$

leaving both δ' and σ requiring their right-hand operands.

Note that, by the commutativity of $\oplus$ and the distributivity of $\odot$, it may be shown that Eq. (3) implies that the operation of Eq. (4) preserves the normalization of Eq. (2). If $\lambda s \odot \delta s = 1$, then:

$$\begin{aligned} \lambda s \odot \delta' s &= \delta p \odot p\sigma s \odot \lambda s \\ &= \delta p \odot (p\sigma s \odot \lambda s) \\ &= \delta p \odot \lambda p \\ &= \delta s \odot \lambda s = 1. \end{aligned}$$

Equations (2), (3) and (4) are a common formal expression for the normalization of hyperstates, the normalization of the NSF, and the transition of hyperstates in all four generalizations of automata so far discussed. The truth-set V, and the operator, $\oplus$, have already been defined for each case. It remains only to define the operator, $\odot$, as: arithmetic $\times$ (multiplication) when $\oplus$ is arithmetic $+$ (deterministic and stochastic automata); and arithmetic **MIN** (least of two operands) when $\oplus$ is **MAX** (fuzzy and non-deterministic automata). Note that **MIN** and **MAX** may be regarded as Boolean **AND** and **OR** in the non-deterministic case, and as generalized multivalued logic **AND** and **OR** in the fuzzy case.

Before proceeding it would probably be useful to illustrate the relation between the notation adopted here and the conventional vector/matrix notation for automata. Suppose that S contains n states labelled $s_1 \ldots s_n$, and that $\oplus$ and $\odot$ are written as addition and multiplication, respectively. Let:

$$\delta s_i = p_i, \delta' s_i = p'_i \tag{6}$$

and:

$$s_j \sigma s_i = p_{ij} \tag{7}$$

Then Eq. (2) becomes:

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} p'_i = 1 \tag{8}$$

Eq. (3) becomes:

$$\sum_{i=1}^{n} p_{ij} = 1 \tag{9}$$

and Eq. (4) becomes:

$$p'_i = \sum_{j=1}^{n} p_{ij} p_j \tag{10}$$

2.8 Comparisons and Contrasts

The previous sections have been phrased to bring out the similarities and differences between the four structures considered. Note that the normalization condition is uniformly that of Eq. 2, and the truth sets are either the entire interval, [0, 1], or its boundary points, {0, 1}, whilst the transitions are uniformly represented by Eqs. (3) and (4). A table of operators against truth sets (Table I) shows that the four cases analysed encompass a *complete* set of variations for these truth sets and operators. This is intuitively satisfying because it gives a closure over those automata which have been most extensively studied in the past. It is an answer in this context to the question of whether we can continually invent new forms of automaton.

TABLE I

Truth sets and operators for automata

		Truth Set {0, 1}	Truth Set [0, 1]		
$\oplus$	+	Deterministic	Probabilistic	×	$\odot$
$\oplus$	OR	Nondeterministic	Fuzzy	AND	$\odot$

2.8.1 Fuzzy and stochastic automata. The relationship expressed in Table I between fuzzy, nondeterministic and deterministic automata, and between stochastic and deterministic automata, are well known. However, that between stochastic and fuzzy automata is less obvious and it is worth discussing whether this is just a mathematical formality or whether it has a semantic content. Clearly the common use of the interval [0, 1] corresponds to quite different interpretations of the values within it—a "degree of membership" appears as a less precise concept than a "probability". Equally the operators, + and ×, appear little related to **MAX** and **MIN**. However, the following argument demonstrates a closer correspondence than might be expected.

Consider two events, A and B, with respective probabilities of occurrence, p_A and p_B. If the two events are statistically independent then the probabilities of their conjunction and disjunction are:

$$p(A \wedge B) = p_A \times p_B \tag{11}$$

$$p(A \vee B) = p_A + p_B - p_A \times p_B \tag{12}$$

Suppose, however, that A and B are not independent events but that one implies the other, $A \rightarrow B$, say. Then we have:

$$p(A \wedge B) = p_A \tag{13}$$

$$p(A \vee B) = p_B \tag{14}$$

However, the direction of implication also gives us:

$$p(A) \leqq p(B) \tag{15}$$

so that Eqs. (13) and (14) may be re-written.

$$p(A \wedge B) = \mathrm{MIN}(p_A, p_B) \tag{16}$$

$$p(A \vee B) = \mathrm{MAX}(p_A, p_B) \tag{17}$$

Conversely, if the "fuzzy logic" conditions of Eqs. (16) and (17) hold for two probabilistic events, then we have:

$$p(A \wedge B) = \mathrm{MIN}(p(A \wedge B) + p(A \wedge \bar{B}), p(A \wedge B) + p(\bar{A} \wedge B)) \tag{18}$$

which implies that either $p(A \wedge \bar{B}) = 0$ or $p(\bar{A} \wedge B) = 0$, i.e. either $A \rightarrow B$ or $B \rightarrow A$.

Thus we see that the applicability of the fuzzy logic operations of Eqs. (16) and (17) to determining the probabilities of conjunction and disjunction of two probabilistic variables is equivalent to their being a logical relationship of implication between the variables. In principle therefore the fuzzy logic connectives may be regarded as those of probabilistic logic in which all variables are connected by a chain of implication. This is the converse condition to that generally assumed in application of probability theory where one attempts to make variables statistically independent. The "chain" concept is intuitively significant—the MIN operation in fuzzy logic expressing that a chain is as weak as its weakest link—the **MAX** operation expressing that alternative chains in parallel are as strong as the strongest.

These relationships between probabilistic and fuzzy logics indicate that Table I expresses more than mathematical formalism. Clearly the relationship demonstrated between fuzzy and probabilistic logics should also extend to the richer semantics developed by Zadeh.[17,18,19] It would also be interesting for application studies to compare probabilistic and fuzzy logics in their relative efficacies for particular situations and relate this to the presence or absence of implications between the variables involved. Gaines[27] has done this for the control studies of Mamdani and Assilian,[23] showing that in this particular case both logics lead to the same control policy. This is clearly not necessarily true in general but might be almost universal in practical situations where the algorithms have to be robust against errors and imprecision in the data, and hence also to *reasonable perturbations in the operators on which they are based.*

2.9 *The Logics of Conventional Automata*

For fuzzy and non-deterministic automata the operators $\oplus$ and $\circ$ are logical **OR** and **AND**, in two-valued or multi-valued logics respectively. It might be expected that the same would be true for deterministic automata since the truth set is two-valued. However, it has already been noted in section 2.3 that $\oplus$ must be arithmetic + in this case if the normalization of deterministic hyperstates is to be expressed in terms of it. Hence, in terms of operators, deterministic automata are more closely related to probabilistic automata than to either non-deterministic or fuzzy automata.

However, there is also a sense in which $\oplus$ and $\circ$ for deterministic and probabilistic automata may be regarded as logical operators. Rescher (Ref. 20, section 27) has given a set of postulates for what he calls a *probability logic* (PL) over a domain of statements. The logic is defined in terms of a valuation over the lattice of conjunction and disjunction of statements that assigns some real value, $P(A)$, to every member, A, of the universe of statements. This assignment has to satisfy the postulates:—

($P1$) $0 \leqq P(A)$, for any statment, A.

($P2$) $P(A \vee \bar{A}) = 1$

($P3$) $P(A \vee B) = P(A) + P(B)$, provided A and B are mutually exclusive

($P4$) $P(A) = P(B)$, if A is logically equivalent to B

($P5$) $P(A \wedge B) = P(A)+P(B)-P(A \vee B)$, defining conjunction

($P6$) $P(A \supset B) = P(\bar{A} \wedge B)$, defining implication

($P7$) $P(A \equiv B) = P(A \supset B \wedge B \supset A)$, defining equivalence

These are the normal basic requirements for a system of probability, but they may also be regarded as a set of postulates for an infinite-valued logic. The logic is not truth-functional, but if the value 1 only is designated then the truth tables for the operations of negation, conjunction and disjunction are those of the classical propositional calculus (PC). Conversely, the axioms that define PC may be shown to be tautologies of probability logic (Ref. 20, p. 187). Hence the system coincides completely with PC in its tautologies.

In terms of our previous discussion of automata the operator $\oplus$ is used to combine the truth-values of the automaton being in different states, or to combine the truth values of trajectories to the same state, both of which represent the disjunction of mutually exclusive events. Hence $P3$ of PL applies and $\oplus$ is arithmetic plus. The operator $\circ$ represents a normalization-preserving transformation of hyperstates and must be arithmetic $\times$ if $\oplus$ is arithmetic $+$. However, it is consistent with $P5$ for the logical conjunction of certain statements to have a valuation which is the product of the valuations of each statement. This condition represents *statistical independence* between the statements.

Thus, for all four cases the operators $\oplus$ and $\circ$ may be regarded as logical operators of disjunction and conjunction respectively. For non-deterministic automata the two-valued propositional calculus is appropriate. For fuzzy automata a multi-valued generalization of PC is appropriate, e.g. virtually any of those described in Ref. 20 since the MAX and MIN operators are the most common for disjunction and conjunction. For deterministic and probabilistic automata a probability logic is appropriate in which $\circ$ plays the role of conjunction for statistically independent statements.

We can make the common basis for automata even stronger now by re-interpreting the arguments of section 2.8.1 in terms of the postulates of PL. Essentially what we have shown in section 2.8.1 is that the operations on truth-values for disjunction and conjunction in a PL become MAX and MIN respectively if a relation of mutual implication between statements is assumed (rather than statistical independence). Hence PL is a general foundation for all four types of automata, with auxiliary postulates leading to the particular cases. This foundational role for PL will be further demonstrated in the following section.

3. POSSIBLE AUTOMATA

3.1 The Need for Further Automaton Structures

Although section 2 gives a satisfying completeness result for the conventional spectrum of automata, it in no way implies the sufficiency of these structures to represent all possible cases of interest. That they are in fact inadequate is best seen by example, and we shall give one which is itself of particular interest in the context of calculi of possibility and probability, and of multi-valued logics.

In our studies of system stability and control we have been very concerned to embody in our formulation the distinction between possible events that *may not occur* and possible events that are *guaranteed to occur* sooner or later. The former events correspond to problems that may arise and have to be avoided. They relate to regions of states which are reachable in terms of stability analysis but not reachable in terms of control. The second type of possible event, however, is responsive to feedback control since if the situation is continually recreated in which it may occur then it eventually *will* occur.

Note that probability theory does not provide an explicatum of the first type of possible event. If for the purposes of analysing an uncertain system we assign an uncertain event a non-zero probability then we imply that not only may it occur but also, in a sequence of occurrences each of which may be that event, it eventually *will* occur with a probability arbitrarily near one. The notional assignment of a definite probability to an event also fails to provide an adequate explicatum of the second type of possible event because it has the stronger implication that the relative frequency of such events in a sequence will tend to converge to the given probability with increasing length of sequence.

Either or both of these connotations which probability has over possibility may be too strong in practical situations where the concepts of probability theory are being used to express the effects of uncertain behaviour. For example, we are often faced with situations where an event, E *may* occur, but there is no guarantee that E actually *will* occur, no matter how long we wait. If we ascribe some arbitrary probability to E then we certainly express that it is a possible event. However we are in a position to derive totally unjustified results based on the certainty of some eventual occurrence of E, or meaningless numeric results based on the actual "probability" of occurrence of E.

The danger of deriving profound results that have no justification other than an unwarranted strength in the theory is a real one. For example, Gaines[28, 29] has shown that a two-state stochastic automaton can solve a class of control problems otherwise requiring a recursive automaton[30] and not soluble by any finite automaton[28, 30]. This significant result is dependent on a source of uncertain behaviour that is properly probabilistic, but whose probability does not have to be known. It cannot be derived if the behaviour is merely possibilistic. There is no way, however, of preventing the consequences of this result appearing in the analysis of a system in which uncertainties have been represented by probabilities rather than possibilities.

A similar problem arises in the practical application of linear systems theory. There are many results which may be derived from the assumption of linearity (such as the complete extension of knowledge of local behaviour to that of global behaviour) which are false in most practical systems. The engineer resolves these problems in practice through a set of "rules-of-thumb" based on commonsense and experience which constrain the deductions he is prepared to assume valid. Such a resolution is however extremely difficult to implement in an automated, or computer-aided, design system, and becomes increasingly difficult to apply as the system involved becomes more complex.

In the following section we analyse these different forms of uncertainty about system behaviour and then demonstrate that whilst any one of them may be encompassed by the automaton structures analysed in section 2 a combination of different forms of uncertainty requires a more general structure than is available in this spectrum of automata.

3.2 *Possible, Eventual and Probable Events*

(i) *Possible* Event E is possible—no reliance may be placed upon the occurrence or the non-occurrence of E. This corresponds to an interpretation of E as an event whose negative consequences must be taken into account, but whose positive consequences cannot be relied upon. The modal operator of "possibility", M, in alethic modal logic[31, 32] represents this concept, but conventional probability theory provides no explicatum for it.

(ii) *Eventual* Event E will eventually occur in that it is *frequent* in the sense of infinite sequences, i.e. in a series of events $E(i)$, for any n, there exists $m > n$, such that $E(m) = E$. This corresponds to the interpretation of E as an event whose eventual occurrence may be relied upon, but whose relative frequency of occurrence is not necessarily stable or known. A suitable explicatum in probability theory is that $p(E) > 0$, the event has a non-zero probability of occurrence (the foundations of probability theory in terms of computational complexity[35, 36] show that any apparent philosophical distinction between "frequent" events and those of "non-zero probability" has no operational interpretation).

(iii) *Probable* Event E is frequent and its relative frequency of occurrence in a sequence of events converges to a definite value, $p(E)$, its probability of occurrence. This is the type of event with which we are most used to dealing using the methods of probability theory.

One approach to incorporating these three forms of uncertainty into a single logic has been suggested by Gaines and Kohout,[33] who have shown that it is possible to take these three types of event and add to them two further types, necessary and impossible events (always or never occur, respectively), to form a multi-valued logic. The logic is mixed discrete-continuous since probable events are represented by a number in the semi-open interval (0, 1]. This approach is outlined in the next section.

3.3 *A Logic of Possibility, Eventuality and Probability*

Let us take the truth set, V, to consist of the semi-open interval, $R \equiv (0, 1]$ and the elements, N, E, P, I, whose interpretation is:

N—Necessary occurrence—probability equals unity.

E—Eventual occurrence—probability unknown but non-zero.

P—Possible—cannot say that it will not occur.

I—Impossible—cannot occur.

A truth value in R is a known probability of occurtence which is not zero. We shall say an event is of type R if its truth value is in R and will write $R:p$, where p is its probability, to emphasize this.

The $\oplus$ operator over V corresponds to two different routes arriving at the same state—what can we say if we know either x or y is true? A truth table for $\oplus$ is given in Table II. The $\odot$ operator over V corresponds to a state followed by a transition—what can we say if we know that y follows x. A truth table for -- is given in Table III.

TABLE II
Truth table for $\oplus$

$\oplus$	N	E	$R:r$	P	I
N	N	N	N	N	N
E	N	E	E	E	E
$R:r'$	N	E	$R:r+r'$	E	$R:r'$
P	N	E	E	P	P
I	N	E	$R:r$	P	I

TABLE III
Truth table for $\odot$

$\odot$	N	E	$R:r$	P	I
N	N	E	$R:r$	P	I
E	E	E	E	P	I
$R:r'$	$R:r'$	E	$R:rr'$	P	I
P	P	P	P	P	I
I	I	I	I	I	I

Consider first the structure with R taken as a single logic variable, i.e. $V \equiv \{N, E, R, P, I\}$, which allows for all the explicata of uncertainty developed in section 3.2. Note that, without R, the tables for $\oplus$ and $\odot$ are simply those of a 4-value Post algebra, and hence can be mapped onto a fuzzy logic. R, however, behaves anomalously in that $R \oplus P = E$ whereas $R \odot P = P$. It has been suggested by Brown[34] that the V-set of a fuzzy logic be taken to be a distributive lattice. However the interaction of R and P is inconsistent with $\oplus$ and $\odot$ being lattice operations. This is a concrete example of the need for more general truth sets discussed by Goguen.[15]

If we now consider the full truth set at first specified in which R is actually a semi-open interval, then the logic is now a mixed continuous discrete structure which can, however, still be neatly represented in the "truth tables". Such structures are both theoretically interesting and practically necessary to obtain rich enough explicata of the behaviour of uncertain systems.

It will be noted that the diagonals of the two tables show the idempotency of the elements, and the wider significance of this may be raised. However, the individual elements of R are clearly not idempotent in general ($p+p \neq p$, and $p \times p \neq p$, in general), and if we consider a variant on E, such as G interpreted as "properly probabilistic" (unknown probability in the open interval, (0, 1)), then idempotency can be seen to fail even for a discrete element ($G \oplus G \neq G$).

Thus this multivalued logic of possibility and probability illustrates the requirement for automaton logics beyond those discussed in section 2. In the following section we shall consider the fundamental constraints upon more general logics. However, having introduced the examples of this section, it is appropriate to briefly summarize further studies of mixed probability, eventuality and possibility.

3.4 *The Problem of Possibility*

The multivalued logic of section 3.3 provides an improved account of the various forms of uncertainty and their mixtures, false conclusions to be drawn about possibilistic or eventualistic events, and yet it contains a full account of truly probabilistic events. However, it suffers from what appears to be a fundamental defect of all attempts to account for possibilistic, or non-deterministic, behaviour in terms of a finite-valued logic. It is unable to sustain certain forms of deduction leading to deterministic conclusions about non-deterministic behaviour.

The problem of drawing conclusions about possible events is best seen in terms of an example. Consider the nondeterministic automaton of Figure 1—starting in $S0$, its future states are indeterminate. However, even if we know only that the transitions are possible, it is clear that the state $S2$ will certainly be entered at some time. If we know also that the transitions are eventual then it is also certain that the ultimate state will be $S5$. If, in addition, the transition probabilities are well-defined then we may also derive the expected time for this state to be reached. This last conclusion is a numeric result readily represented in probabilistic terms, but what of the weaker results? They are not in themselves quantitative but they do seem to be based on an underlying quantitative argument—when the state will be $S2$ is uncertain but the "total uncertainty" about that state sums to a certainty that it will occur.

The normal representation of a non-deterministic transition by a binary logical variable taking the values 0 (impossible) and 1 (possible) cannot be used to support this form of reasoning. For example, Table IV shows the possibility of each

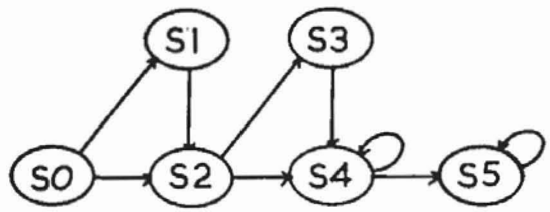

FIGURE 1 State transitions of a non-deterministic automaton.

TABLE IV
Binary form of nondeterministic hyperstates

Time	0	1	2	3	4	5
$S0$	1	0	0	0	0	0
$S1$	0	1	1	0	0	0
$S2$	0	1	1	0	0	0
$S3$	0	0	1	1	0	0
$S4$	0	0	1	1	1	1
$S5$	0	0	0	1	1	1

state of the automaton of Figure 1 at successive clock times. It can be seen that the pattern of behaviour for $S2$ is identical to that for $S3$, and yet we can see that $S2$ must occur whilst $S3$ may only possibly occur. Claerly an exhaustive enumeration of all possible paths from $S0$ to $S5$ will show that $S2$ is on all of them whilst $S3$ is not, but such combinatorial searches become difficult when the system is complex and contains loops (leading to an infinite number of possible paths).

If the transitions were probabilistic the argument could be based on a simple numeric calculation of the total probability of each of the states, $S2$ and $S3$. What appears to be lacking in the binary representation of possible transitions is the normalization possible with probabilities that expresses that the automaton is actually in one, and only one, state. The normalization of the columns of Table 1 is appropriate to a non-deterministic automaton (section 2.6) in that at least one of the states has the value 1, but there is also the auxiliary rule that if only one of the states has the value 1 then the automaton is definitely in that state.

It is in the form of this auxiliary rule that the weakness of expressing possibility in a finite-valued logic seems to lie. To find out if the automaton is definitely in a state we have to examine the possibilities of all other states and show that they are zero. This global argument contrasts sharply with the local reasoning in the probabilistic case that the automaton is definitely in a state because the probability of that state is 1. There seems no reason, however, why we should not retain this "conservation law" so readily expressed in probabilities without giving the actual numeric probabilities anything more than a possibilistic interpretation, i.e.:—

($D1$) $p(E) = 0$ E is impossible

($D2$) $0 < p(E) \leqq 1$ E is possible

($D3$) $p(E) = 1$ E is necessary

A calculus of possibility based on these definitions is quite simply developed and in fact gives non-deterministic automata the structure of probabilistic automata with the weakened semantics that, apart from 0 and 1, the values of "probability" have no greater significance than that an event is possible.

A formal proof that a probability logic may be used as a proper basis for a logic of possibility has been given by Rescher[37] (see Ref. 20, section 28.2). He introduces modalities into the PL of section 2.9 by the stipulations:

($P8$) Necessity: $\mathbf{L}A = 1$ or 0 according as $P(A')$ is, or is not, uniformly 1 for every substitution instance, A', of A.

($P9$) Possibility: $\mathbf{M}A = 0$ or 1 according as $P(A')$ is, or is not, uniformly 0 for every substitution instance, A', of A.

The use of the concept of substitution instances is necessary because the logic is not itself truth-functional. Rescher[37] has demonstrated that the logic with these modalities is characteristic of Lewis' system S5 of modal logic[31] in that its tautologies are precisely those of S5, and vice versa. Thus, whilst there is no finite-valued logic that represents precisely the alethic model logic of necessity and possibility, this (infinite-valued) "probability logic" does so.

If we consider only mutually exclusive events, such as an automaton being in one or another of its states, then it may be seen from $P3$ that the logic becomes truth-functional. Valuations are then just additive over the disjunction of events. Hence also

$P9$ may be interpreted as, "an event is possible if and only if its valuation is non-zero", which may be seen as a binary evaluation similar to the 0/1 representation of impossibility/possible in nondeterministic automata. However we also have the new rule based on $P8$ that, "an event is necessary if its valuation is unity". This corresponds to our previous additional rule that if the disjunction of a set of mutually exclusive events is necessary, and only one of the events is possible, then that event must be necessary. This is now derivable from the purely arithmetic effect of the additivity of positive valuations, i.e. if the sum of a set of numbers is 1, and all but one of those numbers is zero, then that number must be 1.

It is interesting to compare this with the corresponding rule of the modal logic S5 (T28 in Ref. 31, p. 51) that:

$$L(A \vee B) \supset (LA \vee MB)$$

which clearly extends to multiple events:

$$\mathrm{L}(A \vee B \vee C \vee \ldots) \supset (\mathrm{L}A \vee \mathrm{M}B \vee \mathrm{M}C \vee \ldots)$$

i.e. if it is necessary that at least one of a set of events occur then either one of the events is necessary or some of the others are possible. Hence, from the impossibility of all but one event we can infer the necessity for that event. It can be seen that what the 'probability logic" of S5 does is replace a process of logical deduction with one of arithmetic. The failure of a binary representation of possibility to do this may itself be seen as a demonstration of the impossibility of characterizing S5 with a finite-valued logic.[38]

Thus a correct model of the behaviour of a possibilistic automaton may be based on what is effectively a probabilistic automaton with the weakened interpretation of "probabilities" given by $D1$ through $D3$. A re-analysis of the automaton of Figure 1 using this interpretation shows that the difference between states $S2$ and $S3$ that was previously obscured is now apparent. Table V is the new version of Table IV. To show the generality of the result symbols have been used rather than numbers—a and b are any numbers in the open interval, (0, 1). The final column gives the sums of the elements in each row. For $S0$ through $S3$, since the automaton being, for example, in $S2$ at time 1 and at time 2 are mutually exclusive possibilities, the sum properly represents the total possibility of the automaton being in the state. It can be seen that $S1$ and $S3$ are only possibly entered but that $S2$, for which the total is 1, will be necessarily entered. The sums for $S4$ and $S5$ are not meaningful because the loops in the state diagram rule out mutual exclusion and hence the additivity of possibilities.

The penultimate column of Table V shows the final possibility of the automaton being in each of its states. Whilst that for $S4$ is asymptotic to 0 and that for $S5$ is asymptotic to 1, both are essentially

TABLE V

PL Form of nondeterministic hyperstates

Time	0	1	2	3	4	Final	Total
$S0$	1	0	0	0	0	0	1
$S1$	0	a	0	0	0	0	a
$S2$	0	$1-a$	a	0	0	0	1
$S3$	0	0	$b(1-a)$	ba	0	0	d
$S4$	0	0	$(1-b)(1-a)$	>0	>0	$\to 0$	—
$S5$	0	0	0	>0	>0	$\to 1$	—
Total	1	1	1	1	1	1	—

non-zero for all time and hence, if the transitions are possibilistic, the most we can say is that both states are ultimately possible. This serves to illustrate an essential distinction between the analysis of possible and eventual behaviour since, if the transitions are eventual, we may show[39] that an asymptotic approach of the possibility of an event to unity indicates that that event must ultimately necessarily occur.

This section has demonstrated the role of an appropriate interpretation of PL as a full logic of possibility that allows all, and only, those conclusions to be drawn about the behaviour of a possibilistic system that are justified by its semantics. Gaines[39] has also given a suitable interpretation for PL to be a full logic of eventualistic systems as noted in the last paragraph. Thus again, as noted in section 2.9, the probabilistic automaton model seems to have a central role in system modelling in that it subsumes all others. However, whilst a PL based on scalar probabilities in the interval [0, 1] is an adequate basis for any one of the three logics of possibility, eventuality and probability, it is inadequate to account for the behaviour of mixed systems involving any two, or all three, types of event. It can be shown that a *vector probability*[39] is both necessary and sufficient to account for the behaviour of such systems, with $\oplus$ being conventional vector addition but $\odot$ being a rather strange form of vector "multiplication".

However, within the main terms of reference of this paper, the discussion of this section has served to indicate that the conventional spectrum of automata analysed in section 2 is not adequate to provide models for all systems of practical interest. The following section reverses the approach and considers the most general form possible for an automaton.

4. THE GENERAL CASE

In this section we shall draw on the arguments and examples of sections 2 and 3 to develop the most general form of automaton structure that is consistent with the notions of state, and state-determined. The final part of the section is concerned with the other general theoretical questions such as topological models of automata and the role of inputs and outputs.

4.1 *Semirings*

We have noted in section 3.3 that the truth-set need not be a fuzzy set or a distributive lattice, and that the elements need not be idempotents under $\oplus$ or $\circ$. In the example of the previous section it can be seen that $\oplus$ and $\circ$ are both associative and commutative and that $\circ$ distributes over $\oplus$, i.e. together they give the truth set the structure of a commutative semiring. It is also apparent that this semiring is positive[40] (p. 125) in that if we consider the zero element (I in Tables II and III) then:

$$a \oplus b = I \rightarrow a = I = b \qquad (19)$$

and:

$$a \circ b = I \rightarrow a = I \text{ or } b = I \qquad (20)$$

The example of section 3.3 shows that a stronger structure would be too restrictive. However, the question remains of whether a positive commutative semiring is still too strong a structure on which to base automata theory. The following notes outline arguments to show on fundamental, and intuitively satisfying, grounds that at least an ordered semiring is necessary.

First consider the operator, $\oplus$, which represents the combination of different trajectories to the same state. Trajectories may be combined in pairs so that this gives the truth set the structure of a partial groupoid (partial because some pairs of values may not arise and hence their result is undefined, e.g. probabilities of 1 and 1). However, we must also take into account the independence of trajectories, that they represent alternative paths and there should be no effect of order or grouping when combining them. This implies that $\oplus$ is necessarily commutative and associative, and hence defines a partial commutative semigroup over the truth set (it may be taken as a partial monoid by adding the null trajectory as an identity element). We may drop the term "partial" in general by noting that the "don't care" conditions can always be fitted in to complete the monoid.

Even these constraints do not fully represent the necessary structure since each trajectory termination in a state can only *add* to our knowledge about the automaton being in that state. There can be no cancellation of information obtained by considering independent trajectories. One possible expression of this is to require the monoid to be positive, so that:

$$a, b \in V, a \oplus b = 0 \rightarrow a = 0 = b \qquad (21)$$

where 0 is the identity element of the monoid written additively. It can readily be seen (by adding a or b to each side of the left equation of 14) that if the elements of the monoid are idempotent Eq. (14) automatically holds. Idempotency also implies the natural order on the monoid is a semilattice. That is defining a relation, $\geqq$, on V in terms of $\oplus$:

$$a, b \in V, a \geqq b \leftrightarrow \exists c : a = b \oplus c \qquad (22)$$

Unfortunately the positivity condition of Eq. (21) alone does not guarantee that this is even a partial order, and it seems that the best statement of the constraint upon the monoid is that the natural preorder on it defined by Eq. (22) is actually a partial order. This itself implies that the monoid is positive and is implied if the elements are idempotent. Intuitively, this order relation corresponds to our having two independent sources of information about a state which cannot cancel—taken together they must give at least as much information as either alone.

The operator $\circ$ presents more interesting problems since it represents the interaction between states and transitions, and there is no a priori reason to suppose that they can be expressed in the same language. Let us start with the more general assumption that the transitions are drawn from a set of functions, $F \equiv \{f: V \rightarrow V\}$. Considering the same argument as for $\oplus$, it can be seen that the result of applying a function to each individual trajectory separately (and then combining them) must be the same as applying it to them already combined—i.e. the functions must distribute over $\oplus$:

$$f \in F, a, b \in V, f(a \oplus b) = (fa) \oplus (fb) \qquad (23)$$

The implications of distributivity are not intuitively obvious and they may be expressed more meaningfully in terms of the order relation of Eq. (22), since Eq. (23) shows that f must be isotone with respect to $\geqq$. Again we may argue that a transition cannot in itself increase information about a state so that f must be isotone non-increasing.

The isotone non-increasing mappings over the truth set clearly form a semi-group which can be extended to be a semiring by the definitions:

$$f, g, h \in F, h = f \circ g \leftrightarrow a \in V, ha = gfa \quad (24)$$

and:

$$f, g, h \in F, h = f \oplus g \leftrightarrow a \in V, ha = fa \oplus ga \qquad (25)$$

The partial order defined by Eq. (22) has a natural extension to F in terms of Eq. (25) and this in turn implies that F under $\oplus$ and $\circ$ is a positive semiring.

There are a number of possible monomorphisms injecting V into F, $\mu: V \to F$, such that:

$$a \in V, f \in F, \mu(fa) = \mu(a) \circ f \qquad (26)$$

and:

$$a, b \in V, \mu(a \oplus b) = \mu(a) \oplus \mu(b) \qquad (27)$$

i.e. μ imbeds V into the endomorphism semiring in such a way that the $\oplus$ and $\circ$ operators have a common interpretation in both structures. It is the existence of such imbeddings that enable us to use a common language to describe both hyperstates and their transitions.

It will be noted that the examples given previously are such that $\circ$ is commutative whereas no informal arguments have been put forward here to suggest that this is true in general. It is easy enough to generate simple structures in which $\circ$ is not commutative but all our other requirements are satisfied. We have yet to find a semantics for such structure to show that they are necessary. Conversely there appears to be no argument on the lines of those advanced to suggest that such a semantics is not possible.

4.1.1 The role of idempotency If one accepts the informal arguments of the previous section in terms of the monoid over V representing "information" about the automaton being in a state then it would be natural to assume that its elements were idempotents, i.e. that getting the "same" information a second time contributed nothing extra. Only the probabilistic case gives a counter-example, and here the "information" is a value rather than a datum.

Suppose however that instead of considering the probabilities themselves one considers the underlying Borel set structure of the σ-algebra for the probabilities. Then the "information" consists of disjoint sub-sets whose measures correspond to the probabilities and if $\oplus$ is regarded as the union operation on the sub-sets it is, of course, idempotent.

In this case our semiring becomes a lattice, as it was for all the non-probabilistic examples given. Thus it might well be that an intuitively satisfying axiomatization of automata theory could lead to the stronger structure of a lattice, rather than a semiring, providing one is prepared to carry the full structure of a measure algebra when carrying results over to probabilistic automata.

This suggestion throws further light on the relationship between fuzzy and probabilistic automata. The normalization conditions are the same in that the joins of the truth values for all the states should be units, but the fuzzy truth values must form a linearly-ordered chain (a "vertical" section), whereas the probabilistic truth values must form a totally unordered set (a "horizontal" section) whose meets are zero.

4.2 *Topological Models of Logics and Automata*

The remarks of section 4.1.1 suggest that one should examine topological models of automata rather than purely logical models. There is of course a close relationship between multivalued and modal logics and general topologies[41] that has been developed as a powerful tool in logical studies.[42,43] For example, we may employ an **AIOU**-topology[41] as a semantic model of the S4 modal logic.[44]

Some more recent results in automata theory indicate the way in which these relationships may be further exploited. On the one hand, we have theories using metalanguages based on the propositional calculus (PC) for the synthesis and analysis of automata,[45] with, on the other hand, various semantic theories for a variety of types as analysed in this paper. In the first case we have a model $\mathbf{L}_{\mathbf{PC}} \to \mathbf{Aut}$ with statements about automata in $\mathbf{L}_{\mathbf{PC}}$, and with transitions and states as primitives in its model **Aut**. In the second case we have valuations from **Aut** into **V** according to the given type of automata. These are represented diagrammatically in Fig. 2. All sentences which are true in $\mathbf{L_{MOD}}$ must be true in **Aut** and in **V**. In other words **Aut** has to be a model of $\mathbf{L_{MOD}}$, and **V** a model of **Aut**. $\mathbf{L_{PC}}$ has to be extended to a suitable modal language $\mathbf{L_{MOD}}$, taking into account modalities such as possibility, eventuality and necessity. Formal proof techniques may be employed in this research as suggested by Snyder,[32] in a manner similar to that being developed for the logic of protection.[46]

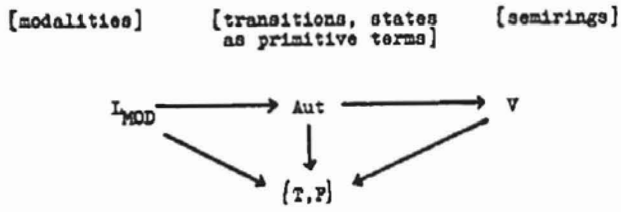

FIGURE 2 Relation between models.

So far we have been concerned with a suitable choice of **V** and with a valuation from **Aut** to **V** for simple automaton structures. The approach may be extended to include less orthodox structures, e.g. $\mathbf{Syst_a}$, describing a particular general systems theory.[47] We may also *compare* primitive terms of various system theories, $\mathbf{Syst_a}$, $\mathbf{Syst_b}$, etc., for a particular valuation and to find a *common fragment* of $\mathbf{L_{MODa}}$ and $\mathbf{L_{MODb}}$, giving the structure shown in Fig. 3. This may be regarded as deriving a formal *analogy relation*[48] between the two systems.

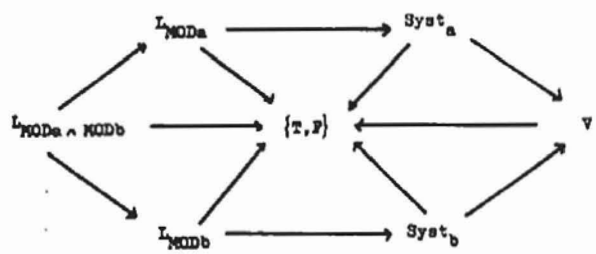

FIGURE 3 Relation between systems.

In order to be able to carry out such comparisons we have to ensure that the middle part of the structure (i.e. **Syst** or **Aut**) is expressed in topological terms compatible with the semantics of modal languages. We have already emphasized[41] the importance of generalized topologies for general sytems theories. It can be shown that the set of all sub-automata of a given automaton forms an **AIOU**-topology. The *accessibility* of a subset of states may therefore be interpreted as possibility in **S4** modal logic. Necessity corresponds to an *interior* in a given **AIOU**-topology,[49] and represents the states which have to be necessarily accessed from a sub-set.

However, the **S4** system of modal logic interpreted in this way does not make any distinction between past and future. A closer look at the **AIOU**-topology of an automaton shows that it contains a further structure which is that of an ordered topological space.[50] It is not therefore surprising to find that the temporal modal logics are fairly subtle refinements of an **S4** system.[51] Prior's work[51] on such logics pays no attention to automata or general systems theory as such and stems from the basic philosophical traditions of Diodorus.[52] However, the open-minded reader will find that Prior's examples may be directly translated into a fundamental discussion of stability, reachability, controllability, etc. in automata and more general systems, and may some day be recognized as a major contribution to this field.

This development in logic has been parallelled or preceded by work examining related *topological* structures, but unfortunately there has been little linking between these two fields. A closer examination of V-sets reveals that we can introduce topologies into it in several different ways, on *two* distinct parts of a V-set. For example, the **AIOU**-topology defining the set of all subautomata of a given automaton is given on the carrier S of the V-set. The operator δ may or may not preserve the topological axioms which hold for S. It is obvious that convergence and limit points in generalised topologies[41] defined on V will play an important role in determining those hyperstates which are probabilistic. It appears that for general automata structures multivalued convergence structures may be necessary.[53]

Development of the theory of sequential spaces aiming at more general models of probability fields opens new pathways in this research. In section 4.1.1 we have already pointed out that axioms for automata structures may lead to a lattice if we are willing to consider σ-sets. A major contribution investigating the relationship between measure theory and convergence spaces (in generalised topologies) is the work of J. Novak (for references see 41, 54, 62). His research on L-spaces[41] aims at the investigation of basic notions of probability,[55] and an interesting parallel with the developments in logic outlined above can be observed. Novak has shown that probabilistic events do not have to be interpreted as subsets (as has been done by Kolmogorov) or elements of Boolean algebra.[56] However, it is essential that they form an algebra which is also a L-space and that the probability is an additive continuous function in L-topology.[57] Related questions of the classification of functions measurable with respect to a particular σ-algebra is considered in Ref. 58. In general, development of the theory of *sequential* spaces, theories of groups and algebras endowed with convergence and of the algebraic operation being continuous with respect to the convergence may play an increasingly important role in general systems and generalised automata research. A fundamental example of a sequential algebra is that of the algebra of subsets of a given set.[59] The group operation is the symmetric difference, the multiplication is the set interesection and the convergence structure is the set convergence of sets (i.e. $A = \lim A_n \Leftrightarrow \liminf A_n = \limsup A_n$).

The generalised topological algebras discussed (with a topology defined with respect to the algebraic operation) pose problems quite distinct from that given by traditional topological groups, semigroups and algebras, so that it is not possible to use the experience of traditional developments or that gained from the development of uniform spaces. This leads to new mathematical problems which may be quite complex, for example, the case given in Figure 1, showing non-additivity in the states $S4$ and $S5$, would suggest that *convergence semigroups* should be developed.

The relationships between languages, logics and topologies have been explored over many years. However, their impact on general systems theory has yet to come. Their importance seems to lie at that half-way house between general mathematical formulations and particular practical applications where the semantics under consideration has itself a high degree of generality.

4.3 The Roles of Inputs and Outputs

This paper has been primarily concerned with the semantics of hyperstates and their transitions, and the roles of inputs and outputs, generally important topics in automata theory, have so far been neglected because neither plays a major role in determining possible automaton structures. Inputs may be described in terms of a mapping from a set of possible inputs to that of legal transition functions, and outputs may be described in terms of a mapping from the set of states (or the product of states and inputs) to the set of possible outputs. However, there are aspects of inputs and outputs that only become apparent within the framework of hyperstate transitions developed in this paper and are not significant for conventional state-determined machines. In this section we will briefly discuss the symmetry between inputs and outputs, and the possibility of representing a hyperstate transition function as due to a hyperinput over normal state transition functions.

In the basic definition of a state-determined machine there appears to be a fundamental asymmetry in the roles of inputs and outputs. The input function plays a major role in determining future behaviour in that the NSF is determined by it. However, the output function is simply a mapping that loses information about the state (we will assume a Moore model in which the current output is a function of the current state only), and since the state itself is well-defined by the previous state and input this "loss" is not a real one—the output function plays no real role and is often assumed to be an identity map.

However, when the previous state, or the input, is uncertain, the next state is not well-defined by the NSF and the output comes to play a far more important role. For example, if the output function is an identity map then even though the predicted next "state" is a non-sharp hyperstate it becomes sharp immediately the output is observed. Thus the output is no longer redundant and in this extreme case it may be used to determine the current state whilst the previous state and hyperinput cannot do so.

In intermediate cases, where the output function is not information-lossless, both the previous input and the current output are relevant to determining the transition from the previous to the current hyperstate. Thus, whilst the deterministic case leads one to regard the NSF as being a function of the input, in the other cases it is best regarded as a function of the input-output pair. This is exemplified in the definition of a system state and in the literature on system identification where a state is defined in terms of the relation between input-output segments and state identification consists of reducing a general hyperstate to a sharp hyperstate by considering the intervening input-output segment.

Hence, in general system-theoretic terms, we may regard the input-output pair as a datum about a system, the input-component of which is at our choice but the output-component of which is not and may only be determined through observation. This viewpoint confounds two components of an automaton structure which are normally regarded as quite distinct. However, this smearing of distinctions is inherent in the very generalization of the concept of an automaton that lead to the definition of a hyperstate since this itself confounds the structural properties of a state-determined machine and the uncertainty of an observer about its exact states and inputs.

It is interesting to demonstrate that this smearing is intrinsic and that no decision-procedure may distinguish between the NSF itself being non-deterministic and the NSF being completely deterministic but the input being imprecisely defined. This is readily seen if, for any state set S, we consider the family of all possible deterministic NSF's over S and regard each one as being selected by some particular input. Now any particular NSF may be decomposed by selecting the least non-zero element in its transition matrix and subtracting out an appropriate deterministic matrix whose corresponding input is weighted according to the value of the element (this can always be done provided the semiring is fully ordered). Repeating this procedure until there are no residual non-zero elements gives the hyperinput corresponding to the NSF.

5 SUMMARY AND CONCLUSIONS

The overall objective of this paper has been to relate the formal mathematical aspects of automata theory to the basic semantics of the notions of state and state-determined systems. In particular we have concentrated on the generalization of the concept of a state/input-determined machine to that of a hyperstate/hyperinput determined machine (section 2.1).

We first explored the conventional spectrum of deterministic/probabilistic/fuzzy/non-deterministic automata (sections 2.3–2.6) and showed that these may be fitted into a single common framework with the same transition and normalization equations (section 2.7) but with the truth set being either the closed unit interval or its end points, and the operation being either arithmetic add/multiply or logical OR/AND (section 2.8). An alternative interpretation of the normalization of fuzzy states was proposed (section 2.4.1), and the relationship between fuzzy and probabilistic automata was shown to stem from a common basis in probability logic (sections 2.8.1 and 2.9).

We then went on to demonstrate that although the conventional spectrum of automata does form a natural set, closed in some sense, it is inadequate to provide models for various forms of uncertain system behaviour encountered in studies of reliability and stability (section 3.1). *Possible*, *eventual* and *probable* events were defined (section 3.2) and a mixed continuous/discrete logic developed for them that did not have the common properties of the logics of the automata already studied (section 3.3). It was also demonstrated that the conventional use of a two-valued logic for non-deterministic automata was inadequate to support certain legitimate arguments about their behaviour (section 3.4), and it was shown that a modalized probability logic was adequate to do this (section 3.5).

Finally, the most general formal structure for an automaton that is consistent with the notions of state and state-determination has been developed (section 4.1). This turns out to be a positive ordered semiring. The remaining sub-sections then explore some further questions raised in passing, that of the relationship between topological and modal logic models of automata (section 4.2) and the role of inputs and outputs in generalized automata (section 4.3).

This has been an exploratory paper with the objective of developing a new approach to automata theory on firm semantic foundations rather than giving a complete formal model with the minimum of justification. General systems theory needs an armoury of system types which may be used as well-tried weapons to overcome new problems. It is vital that those weapons are well-understood both intuitively and mathematically so that the precise impact of their powers and deficiencies can be weighed in advance. We hope these notes have made some contribution to such an evaluation.

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